Lecture 1. Kinks and domain walls

Consider a quantum field theory defined by the Lagrangian $\mathcal{L}$

$$S = \int d^2x\, \mathcal{L}\,; \qquad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{g^2}{4}(\varphi^2 - v^2)^2.$$

The theory is formulated in two dimensions, i.e. $x^\mu = (t, x)$. Usually we deal with such theories assuming that φ develops a vacuum expectation value v. We then write $\varphi = v + \chi$, where χ is a small fluctuation around v, and study QFT of the field χ. This is spontaneous breaking of Z_2 symmetry $(\varphi \to -\varphi)$ and it leads to the appearance of the "Higgs" boson.

We will now imagine a slightly different situation, namely the case where the field $\varphi(x)$ interpolates between different vacua. Indeed, imagine

$\varphi(x) \to -v$, $x \to -\infty$ and $\varphi(x) \to +v$, $x \to +\infty$. We will see that solutions to field equations exist that

a) satisfy those boundary conditions,

b) have finite energy.

To understand this situation, we will first write the expression for the Hamilton operator:

$$H = \int dx \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{g^2}{4}(\varphi^2 - v^2)^2 \right\},$$

where $\pi = \frac{\partial\phi}{\partial t}$. We are interested in the solutions with minimal energy. This implies $\pi = \frac{\partial\phi}{\partial t} = 0$, i.e. we will study static solutions

The condition $|\varphi| \to v$, $x \to \pm\infty$ is a necessary condition for the solution to have finite energy.

Static equations of motion read:

$$-\frac{d^2\phi}{dx^2} + g^2(\varphi^2 - v^2)\phi = 0. \qquad (*)$$

To solve this equation, we multiply it with $\frac{d\phi}{dx}$ & write $\frac{d\phi}{dx} \cdot \frac{d^2\phi}{dx^2} = \frac{1}{2}\frac{d}{dx}\left[\left(\frac{d\phi}{dx}\right)\right]^2$ and

$$\frac{d\phi}{dx} g^2(\varphi^2 - v^2)\phi \equiv \frac{1}{4}\frac{d}{dx}\left[g^2(\varphi^2 - v^2)^2\right]. \Rightarrow$$

Eq $(*)$ becomes $+\frac{d}{dx}\left[-\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{4}\left(g^2(\varphi^2 - v^2)^2\right)\right] = 0$

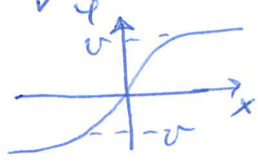
$$\Rightarrow \quad -\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{4}\left(g^2(\varphi^2 - v^2)^2\right) \equiv \text{const}$$

What the integration constant can be?

For the finite energy solution, it should be zero since at $|x| \to \infty$, both $\varphi^2 = v^2$ & $\frac{d\phi}{dx} = 0$. Hence,

$$\boxed{\frac{1}{4}\left(g^2(\varphi^2 - v^2)^2\right) = \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2}$$

We can take the square root to obtain:

$$\frac{d\varphi}{dx} = \frac{g}{\sqrt{2}}\left(v^2 - \varphi^2\right).$$ We choose the sign here ~~following~~ using the following consideration: we consider a solution that changes from $\varphi = -v$ to $\varphi = +v$ as one moves from $x = -\infty$ to $x = +\infty$. Hence everywhere $v^2 - \varphi^2 > 0$ and $\frac{d\varphi}{dx} > 0$. Pictorially 

Now, given the equation for $\frac{d\varphi}{dx}$, the solution is easy:

$$\frac{d\varphi}{v^2 - \varphi^2} = \frac{g\,dx}{\sqrt{2}} \Rightarrow \frac{1}{2v}\left(\ln \frac{v+\varphi}{v-\varphi}\right) = \frac{g(x-x_0)}{\sqrt{2}} \Rightarrow$$

$$\boxed{\varphi(x) = v\,\cancel{\text{atanh}}\,\cancel{\text{cth}}\,\text{th}\left(\frac{vg}{\sqrt{2}}(x-x_0)\right)}$$

The solution, indeed, has desired asymptotics at $x = \pm\infty$. The parameter x_0 is the place where ~~$\varphi(x)=$~~ the field vanishes. ~~It doesn't~~ ~~It corresponds to~~ The existence of this parameter follows from the invariance of the action over shifts. We can use the solutions that we found to compute the energy of the soliton. But we'll do this in a slightly different way.

Let us introduce the "superpotential":
We write $U(\phi) = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2$. Since

$U(\phi) = \frac{g^2}{4}(\varphi^2 - v^2)^2$, we have $\frac{dW}{d\phi} = \frac{g}{\sqrt{2}}(\varphi^2 - v^2)$

$\Rightarrow \quad W = \frac{g}{\sqrt{2}}\left(\frac{1}{3}\phi^3 - v^2\phi\right)$. The energy

reads

$$H = \int dx \left\{ \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2 \right\} =$$

$$= \int dx \left\{ \frac{1}{2}\left(\frac{d\phi}{dx} \pm \frac{dW}{d\phi}\right)^2 \mp \frac{d\phi}{dx}\cdot\frac{dW}{d\phi} \right\} =$$

$$= \int dx \left\{ \frac{1}{2}\left(\frac{d\phi}{dx} \pm \frac{dW}{d\phi}\right)^2 \mp \frac{dW(\phi)}{dx} \right\}$$

$$H = \mp\left[W(\phi(x=+\infty)) - W(\phi(x=-\infty))\right] + \frac{1}{2}\int dx \left(\frac{d\phi}{dx} \pm \frac{dW}{d\phi}\right)^2.$$

Since the integrand is positive definite, the solution with fixed $\phi|_{x=+\infty}$ and $\phi|_{x=-\infty}$ has minimal energy if $\underline{\frac{d\phi}{dx} \pm \frac{dW}{d\phi} = 0}$.

The two signs distinguish between a <u>kink</u> and an <u>anti-kink</u>. For the kink case, we choose the sign "+". Then

$$H_{kink} = -W(\phi(x=+\infty)) + W(\phi(x=-\infty))$$

$\phi(x = \pm\infty) = \pm v$, so that

$$W(\phi(x=+\infty)) = \frac{g}{\sqrt{2}}\left(\frac{1}{3}v^3 - v^3\right) = \frac{g}{\sqrt{2}}\left(-\frac{2}{3}\right)v^3$$

$$W(\phi(x=-\infty)) = \frac{g}{\sqrt{2}}\left(-\frac{1}{3}v^3 + v^3\right) = \frac{g}{\sqrt{2}}\left(\frac{2}{3}\right)v^3$$

Hence, we find

$$\boxed{E_{kink} = \frac{4 g v^3}{3\sqrt{2}}}.$$

We will now discuss some other aspects of the solution. First, the solution that we described is static. We can make it time-dependent by boosting it to another reference frame.

Then $\varphi(x,t|\beta) = v \, \mathrm{th}\left[\frac{vg}{\sqrt{2}} \frac{x - x_0 - \beta t}{\sqrt{1-\beta^2}}\right]$

is a valid solution [see ~~the~~. homework].

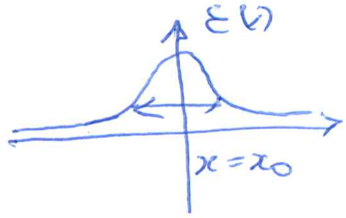
Hence, we get large family of particle-like solutions that can move with finite velocity. To further emphasize its particle nature, we compute the energy density

$$\mathcal{E}(x) = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{g^2}{4}(\varphi^2 - v^2)^2.$$

~~The~~ For the kink, $\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = \frac{g^2}{4}(\varphi^2 - v^2)^2 \Rightarrow$

$$\mathcal{E}(x) = \frac{g^2}{2}(\varphi^2 - v^2)^2 \qquad \varphi^2 - v^2 = \left[\frac{\mathrm{sh}^2(\theta)}{\mathrm{ch}^2(\theta)} - 1\right] v^2 = \frac{-v^2}{\mathrm{ch}^2\theta}$$

$$\Rightarrow \quad \mathcal{E}(x) = \frac{g^2 v^4}{2\,\mathrm{ch}^4\left[\frac{vg}{\sqrt{2}}(x - x_0)\right]} \quad \Rightarrow$$



The region where the energy density is

substantialy different from zero has the size $\lambda_{kink} = \frac{\sqrt{2}}{vg}$; this size is the same (roughly), as the Compton wave length of a particle excitation around symmetric ($\varphi(\pm\infty) \equiv +v$) solution. <u>However, the kink can ~~is~~ be much heavier.</u>

The kink solution can be discussed in the context of topology. For a two-dimensional problem, this is a bit of an overkill, but it is still quite useful for getting an idea about what this is.

Let's consider solutions with finite energies. Those solutions <u>must</u> have asymptotic behavior $\varphi(x)\big|_{x\to\pm\infty} \to \pm v$ or $\mp v$.

Hence, there are 4 options:

1) $x=-\infty \quad \varphi=-v \qquad x=+\infty \quad \varphi=-v$
2) $x=-\infty \quad \varphi=+v \qquad x=+\infty \quad \varphi=+v$
3) $x=-\infty \quad \varphi=-v \qquad x=+\infty \quad \varphi=+v$
4) $x=-\infty \quad \varphi=v \qquad x=+\infty \quad \varphi=-v$.

It is impossible to change from one solution to the other without creating ~~an~~ field ~~[illegible]~~ configuration with infinite energy.

Among 4 solutions, 1) & 2) are simple solutions with x-independent vacuua and oscillations around them. ~~Ther~~ The other solutions, 3) and 4), are kink and anti-kink. Those solutions are topologically non-trivial. Note that the existence of topologically non-trivial sollutions follows from the possibility to have a non-trivial mapping of a spatial infinity onto the vacuum manifold of the theory.
This fact has a general character; it plays a very important role for the existence of non-trivial topological solutions.

We can go one step further and introduce a "topological current". To do so, notice that ~~for~~ in our 2-dim. theory, there is a Levi-Ceritta tensor $\varepsilon^{\mu\nu}$. Let us write $J^\mu \equiv -\varepsilon^{\mu\nu}\partial_\nu W(\phi)$. The current J^μ is conserved: $\partial_\mu J^\mu = -\varepsilon^{\mu\nu}\partial_\mu\partial_\nu W = \equiv 0.$

Note that this current is conserved for all field configurations, i.e. a field, for which $\partial_\mu J^\mu = 0$, doesn't need to be an ~~equation~~ solution of the equations of motion. (as is the case for the Noether currents). ~~Ale~~ Existence of the conserved current implies conserved charge

$$\Phi = \int_{-\infty}^{\infty} dx\, J^0(t,x) = -\int_{-\infty}^{\infty} dx \frac{dW(\phi)}{dx} = -W(\phi_\infty) + W(\phi_{-\infty}).$$

For "trivial" solutions, $\Phi = 0$. For non-trivial,

$$\Phi_{kink} = \frac{4gv^3}{\sqrt{2}\,3} \;; \quad \Phi_{anti-kink} = -\frac{4gv^3}{\sqrt{2}\cdot 3}$$

We will now discuss the stability of the kink solution. Suppose $\varphi_K(\vec{x})$ is the <u>static kink solution</u>. Let's consider small field perturbations around it:

$$\varphi(x,t) = \varphi_K(\vec{x}) + \phi(x,t).$$

The field $\varphi(x,t)$ satisfies the Lagrange equations $-\partial_\mu \partial_\mu \varphi - \frac{\partial V}{\partial \varphi} = 0$. Expanding this equation to first order in $\phi(x,t)$, we find

$$-\partial_\mu \partial_\mu (\varphi_K + \phi) - \frac{\partial V}{\partial \varphi}\Big|_{\varphi=\varphi_K} - \frac{\partial^2 V}{\partial \varphi^2}\Big|(\varphi_K)\, \phi = 0$$

The field φ_K satisfies equations of motion, therefore, the field ϕ satisfies the equation

$$\boxed{-\partial_\mu \partial^\mu \phi - \frac{\partial^2 V}{\partial \varphi^2}\Big|_{\varphi=\varphi_K} \phi = 0}$$

The potential energy computed for the kink field only depends on $\vec{x}$. For this reason, we can search for the solution ϕ

in the following way: $\varphi(t,\vec{x}) = e^{i\omega t} f_\omega(\vec{x})$

Then $$\omega^2 f_\omega + \frac{d^2 f_\omega}{dx^2} - \frac{\partial^2 V}{\partial\varphi^2}\Big|_{\varphi=\varphi_K} \cdot f_\omega = 0$$

We can rewrite it as

$$\boxed{-\frac{d^2 f_\omega}{dx^2} + \frac{\partial^2 V}{\partial\varphi^2}\Big|_{\varphi=\varphi_K} f_\omega = \omega^2 f_\omega}$$

Formally, this can be thought as a Schrödinger equation with the potential

$\frac{\partial^2 V}{\partial\varphi^2}\Big|_{\varphi=\varphi_K}$ which in our case is

$$\frac{\partial^2 V}{\partial\varphi^2}\Big|_{\varphi=\varphi_K} = \frac{\partial^2}{\partial\varphi^2}\left(+\frac{g^2}{4}(\varphi^2-v^2)^2\right)\Big|_{\varphi=\varphi_K} = \frac{\partial}{\partial\varphi}\left(g^2(\varphi^2-v^2)\varphi\right)\Big|_{\varphi=\varphi_K} =$$
$$= g^2(3\varphi^2-v^2)\Big|_{\varphi=\varphi_K} = g^2\left(3v^2 th^2\left(\frac{vg}{\sqrt{2}}x\right)-v^2\right) =$$
$$= g^2v^2\left(3\,th^2\left(\frac{vg}{\sqrt{2}}x\right)-1\right).$$

The question of the stability of the solution is the question about negative eigenvalues of the operator $-\frac{d^2}{dx^2} + \frac{\partial^2 V}{\partial\varphi^2}\Big|_{\varphi=\varphi_K}$

If this operator has negative eigenvalues $-\lambda$, then $\omega^2 = -\lambda \Rightarrow \omega = \pm i\sqrt{\lambda}$ and $\varphi(t,x) \sim e^{\sqrt{\lambda}t} f_\omega(x)$, so as $t\to\infty$, the perturbation grows exponentially and the kink is unstable. But, if all eigenvalues are positive, the fluctuation oscillates around the vacuum solution.

There is an <u>interesting feature</u> of this analysis that can be illustrated <u>in full generality</u>. Let's write down the equation for the kink solution itself:

$$\cancel{f_0 f x} \quad -\frac{d^2\varphi_K}{dx^2} + \left.\frac{\partial V}{\partial \varphi}\right|_{\varphi_K} = 0$$

and differentiate it once again. w.r.t. x:

We obtain

$$-\frac{d^2}{dx^2}\left(\frac{d\varphi_K}{dx}\right) + \left.\frac{\partial^2 V}{\partial \varphi^2}\right|_{\varphi=\varphi_K} \cdot \left(\frac{d\varphi_K}{dx}\right) = \phi.$$

It follows from this equation that

$$\phi(t,x) \equiv \frac{d\varphi_K}{dx} \quad \text{is a solution of the}$$

equation that a small perturbation should satisfy with $\omega = 0$. This means that if $\varphi_K(x)$ is a kink solution, then $\varphi_K(x+a)$ is also a solution: $\varphi_K(x+a) = \varphi_K(x) + \frac{d\varphi_K}{dx}\cdot a + \ldots$

A solution with $\omega = 0$ is called <u>a zero-mode solution</u>; the number of zero order solutions & features reflect the symmetries of the original Lagrangian.