Advanced Quantum Field Theory

V: Prof. Kirill Melnikov, Ü: Dr. Lorenzo Tancredi

Exercise Sheet 9

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Problem 1 - Stability of the electroweak ground state in the SM

In class we have studied the case of a simple ϕ^4 theory with a potential characterized by two relative minima, say ϕ_- and ϕ_+, only one of which, say ϕ_-, is an absolute minimum. We have seen that quantum mechanics renders the vacuum ϕ_+ a *false vacuum*, allowing its decay by tunnelling to the real vacuum ϕ_-. The details of the decay were worked out in this particularly simple theory. It is indeed very interesting to figure out whether a similar mechanism can take place also in the electroweak standard model (SM), rendering the electroweak vacuum, and therefore the universe as we know it, unstable under quantum fluctuations. If this was the case, one could use the methods described in the lecture in order to estimate the life-time of the universe, and if this turned out to be much smaller than the age of the universe, one could then argue the necessity of new physics *beyond the standard model* (BSM) in order to stabilize the vacuum we live in.
The SM contains a complex scalar doublet with hypercharge -1, the Higgs field,

$$\phi = \begin{pmatrix} (h + i\,G)/\sqrt{2} \\ G^- \end{pmatrix} \tag{1}$$

with a potential

$$V(\phi) = m^2|\phi|^2 + \lambda|\phi|^4 = \frac{1}{2}m^2h^2 + \frac{1}{4}\,\lambda\,h^4 + ... \tag{2}$$

where we neglected terms that go to zero when $G \to G^- \to 0$. The neutral component H acquires a non-vanishing expectation value $\langle H \rangle = v$, where

$$v = \left(G_F\sqrt{2}\right)^{1/2} \approx 246 \text{ GeV}\,,$$

and the mass of the field h is $m_H^2 = V''(h)\Big|_{h=v} = 2\,\lambda\,v^2$. The SM potential, at variance with the example discussed in class, does not develop *classically* two relative minima of different depth. On the other hand, though, quantum corrections can in principle make the classical vacuum unstable. If we consider field configurations with $h \gg v$, to good accuracy we can approximate the potential as

$$V(h) \approx \frac{1}{4}\,\lambda(h)\,h^4 \tag{3}$$

where $\lambda(\mu)$ is now the running coupling computed at scale μ, which absorbs the quantum corrections to $V(h)$. It becomes than clear that, if for some value of h $\lambda(h)$ becomes negative, then the minima in $h = \pm v$ cannot be absolute minima anymore and the electroweak vacuum becomes unstable. We want to try to calculate a bounce solution in this situation.

1. We know that the bounce solution of minimum action must be invariant under four-dimensional rotations in euclidean space-time, i.e.

$$h = h(r)\,, \qquad r^2 = |\vec{x}|^2 + \tau^2 \quad \text{where } \tau \text{ is the euclidean time.}$$

Assume for simplicity that λ is fixed to some numerical negative value, $\lambda < 0$. Show that the field equation for the bounce becomes

$$h''(r) + \frac{3}{r}h'(r) = \lambda h^3(r)\,. \tag{4}$$

2. We look for a solution that interpolates between the false vacuum at infinite euclidean time and the real vacuum at $\tau = 0$, i.e.

$$h(r)\Big|_{r\to\infty} = v \approx 0\,, \tag{5}$$

where we are neglecting the value of $v \approx 246$ GeV as in Eq. (3). We require moreover that the solution is regular at the origin $r = 0$

$$\frac{dh(r)}{dr}\Big|_{r=0} = 0\,. \tag{6}$$

3. Assuming that $\lambda < 0$, solve Eq. (4) using as Ansatz a series expansion

$$h_0(r) = \sum_{k=0}^{\infty} A_k\, r^k\,, \qquad \text{with} \quad A_0 > 0\,.$$

Show that the solution can be written as

$$h_0(r) = \sqrt{\frac{8}{|\lambda|}}\,\frac{R}{R^2 + r^2}\,. \tag{7}$$

Note that, since

$$h_0(0) = \sqrt{\frac{8}{|\lambda|}}\,\frac{1}{R}$$

we can associate R with the value of the bounce in $r = 0$.

4. Calculate the value of the euclidean action $S[h_0]$ on the bounce solution (7) and show that it is independent of the value of R. This is a consequence of the invariance under scale transformations (i.e. *conformal invariance*).

5. Show that Eq. (4) is invariant under scale transformations, i.e. show that if $h(r)$ is a solution of (4), then

$$h(r) \to a\, h_a(ar)$$

is also a solution.

6. Using the explicit expression for the bounce action derived above and the tunnelling probability in the semi-classical approximation

$$p \approx \left(\frac{T_U}{R}\right)^4 e^{-S[h_0]}$$

estimate the allowed values of λ in function of the parameter R, to be compatible with the actual age of the universe $T_U \approx 10^{10}$ years.

7. In order to gain a quantitative understanding, one must compute the running of $\lambda(\mu)$ in the standard model, which is mainly driven by the value of the top mass, m_t. This has been done by different authors in the last years in order to determine, as precisely as possible, the fate of our universe according to the physics we know. Read papers [1, 2] (and, if you wish, further references therein). Can you figure out how this works? In particular, it is shown there that the typical energy at which λ becomes negative is $\Lambda_I \approx 10^{10}$ GeV. What does this imply for the size of the bounce?

References

[1] G. Isidori, G. Ridolfi and A. Strumia, Nucl. Phys. B **609** (2001) 387 [hep-ph/0104016].

[2] G. Degrassi, S. Di Vita, J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori and A. Strumia, [arXiv:1205.6497].