Advanced Quantum Field Theory

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Exercise Sheet 8

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Problem 1 - A quantum theory of non-topological solitons

In this exercise we want to try and provide a quantum description of solitons as a *quantum coherent state*. We will do this explicitly in the case of a $1+1$-dimensional scalar theory and for a *non-topological soliton*, see for reference [1]. In class we have been dealing so far with so-called *topological solitons*, i.e. soliton solutions which posses a topological charge. In general, it is also possible to find soliton solutions which do not posses a topological charge, i.e. *non-topological* solitons. Consider the following Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^2 - m^2\,\phi^2 + g^2\,\phi^4\,, \qquad \text{with} \qquad m^2, g^2 > 0\,, \tag{1}$$

and $x^\mu = (t, x)\,.$

1. The Lagrangian has a vacuum in $\phi = 0$, which can become unstable for large values of the field. Plot the profile of the potential in function of ϕ. What happens for $\phi = m/g$?
2. Derive the conjugate momentum π and the Hamiltonian density.
3. We want to look for a static solution with finite energy, which interpolates between the values $\phi = 0$ for $x = \pm\infty$ and $\phi = m/g$ for $x = 0$. Show that the E.o.M. satisfied by a static field configuration $\phi_{sol}(x)$ $(\partial_t \phi_{sol}(x) = 0)$ reads
$$\partial_x^2 \phi_{sol} - m^2\,\phi^2 + 2\,g^2\,\phi^3 = 0\,.$$
4. Show that a solution to this equation can be found and reads
$$\phi_{sol}(x) = \frac{m}{g}\text{sech}(m\,x) = \frac{m}{g}\frac{1}{\cosh(m\,x)}\,. \tag{2}$$
5. Show that the energy of this solitonic configuration is
$$E_{sol} = \frac{4\,m^3}{3\,g^2}\,. \tag{3}$$
6. Now expand your classical soliton field in plane waves as follows
$$\phi_{sol}(x) = \sqrt{R}\int \frac{dk}{\sqrt{4\,\pi\,|k|}}\left(e^{i\,k\,x}\alpha_k + e^{-i\,k\,x}\alpha_k^*\right)\,, \tag{4}$$
where the α_k and α_k^* are c-numbers. Prove that they satisfy
$$\alpha_k^*\alpha_k = \pi\,\frac{|k|}{R}\,\frac{1}{g^2}\text{sech}^2\left(\frac{\pi k}{2m}\right)\,. \tag{5}$$

7. We would like to represent our non-topological soliton as a quantum coherent state, $|sol\rangle$. In order to do this we start with identifying $\alpha_k,\, \alpha_k^* \to \hat{\alpha}_k,\, \hat{\alpha}_k^\dagger$ and imposing

$$[\hat{\alpha}_k, \hat{\alpha}_{k'}^\dagger] = \delta_{kk'} \,.$$

First of all, recall that a quantum coherent state is defined to be the eigenstate of the annihilation operator. For a given fixed k, the coherent state $|\alpha_k\rangle$ is defined as

$$\hat{\alpha}_k|\alpha_k\rangle = \alpha_k\, |\alpha_k\rangle \,. \tag{6}$$

Show that the solution to the eigenvalue problem (6) can be written as

$$|\alpha_k\rangle = e^{-\frac{1}{2}|\alpha_k|^2} e^{\alpha_k\, \hat{\alpha}_k^\dagger}|0\rangle = e^{-\frac{1}{2}|\alpha_k|^2} \sum_{n_k=0}^{\infty} \frac{\alpha_k^{n_k}}{\sqrt{n_k!}}|n_k\rangle \tag{7}$$

where $|n_k\rangle$ are the eigenstates of the number operator $\hat{N}_k = \hat{\alpha}_k^\dagger \hat{\alpha}_k$.

8. For every value of the momentum k we can write a coherent state $|\alpha_k\rangle$. Write now the soliton state as a tensor product of coherent states for different k

$$|sol\rangle = \prod_{\otimes k} |\alpha_k\rangle \,. \tag{8}$$

Show that

$$\langle sol|\hat{\alpha}_k|sol\rangle = \alpha_k, \qquad \langle sol|\hat{\alpha}_k^\dagger|sol\rangle = \alpha_k^*,$$

i.e. the the Fourier expansion coefficients of the classical field are the expectation values of the Fock space operators.[1]

9. Define now

$$N_k = \langle sol|\hat{N}_k|sol\rangle$$

and compute the total mean occupation number

$$N = \int_{-\infty}^{+\infty} dk\, N_k \,, \tag{9}$$

and prove that this is a finite number.

10. Compute the amplitude for the production of a soliton state from the vacuum

$$A_{|0\rangle \to |sol\rangle} = \langle 0|sol\rangle \,.$$

Show that for any coherent state of the form (8) one gets

$$\langle 0|sol\rangle = e^{-\frac{N}{2}} \,, \tag{10}$$

where N is the total occupation number of the soliton state. In the case of a non-topological soliton N is finite, see Eq. (9), which implies that the amplitude computed above does not vanish! What does this imply for the vacuum state $|0\rangle$?

11. Finally, compute the energy of this state, defined as

$$E = \int_{-\infty}^{+\infty} dk\, |k|\, N_k \,,$$

and compare it to the classical result (3).

References

[1] G. Dvali, C. Gomez, L. Gruending and T. Rug, arXiv:1508.03074 [hep-th].

[1]Note that these operators do not destroy or create asymptotic propagating quanta but the create and destroy solitonic constituents!