Advanced Quantum Field Theory

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Exercise Sheet 7

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Problem 1 - Dyons: asymptotic behaviour

During the lectures we discussed magnetic monopoles that have zero electric charge. The Georgi-Glashow (GG) model also contains topological solutions that have both a non-zero magnetic as well as non-zero electric charge. These solutions are called *dyons*. A static electric charge can be described by a time-independent non-zero potential $A_0(\vec{x})$. Therefore dyons of minimal energy are static solutions where $A_\mu(\vec{x})$ and $\phi(\vec{x})$ are non-zero and time-independent. The Lagrangian of the model is

$$\mathcal{L} = -\frac{1}{4\,g^2} G^a_{\mu\nu}\, G^{\mu\nu,a} + \frac{1}{2}(\mathcal{D}_\mu \phi^a)(\mathcal{D}^\mu \phi^a) - \lambda\,(\phi^a\phi^a - v^2)^2,$$
$$G^a_{jk} = \partial_j A^a_k - \partial_k A^a_j + \epsilon^{abc} A^b_j A^c_k, \qquad \mathcal{D}_\mu\phi^a = \partial_\mu \phi^a + \epsilon^{abc} A^b_\mu \phi^c\,. \tag{1}$$

1. Show that the energy functional for a solution of the form $A_\mu = A_\mu(\vec{x})$ and $\phi = \phi(\vec{x})$ is:

$$E = \int d^3x \left\{ \frac{1}{2\,g^2} G^a_{0i}\, G^{0i,a} + \frac{1}{4\,g^2} G^a_{ij}\, G^{ij,a} + \frac{1}{2}\left[(A^b_0 A^b_0)\,(\phi^c\phi^c) - (A^b_0\phi^b)^2 \right] + \frac{1}{2}(\mathcal{D}_i\phi^a)(\mathcal{D}^i\phi^a) + U(\phi) \right\}$$

$$U(\phi) = \lambda\,(\phi^a\phi^a - v^2)^2\,. \tag{2}$$

2. In what follows, we assume the same Ansatz for ϕ^a and A^a_i as for the monopole solution, i.e.

$$\phi^a = v\, n^a\, H(r)\,, \qquad A^a_i = \epsilon^{aij} \frac{1}{r} n^j F(r)\,, \quad \text{and} \quad \phi^a_{vac} = v\, n^a\,, \tag{3}$$

with the usual boundary conditions

$$F(r\to\infty) = H(r\to\infty) = 1\,, \qquad F(r\to 0) = H(r \to 0) = 0\,. \tag{4}$$

We need an Ansatz for $A_0(\vec{x})$ as well. We look again for a spherically symmetric soliton solution and try with the *Ansatz*

$$A^a_0 = n^a\, B(r)\,, \tag{5}$$

where $n^i = x^i/r$ and $B(r)$ is a scalar function of r only. How does the Energy functional (2) simplify under this assumption?

3. Use the requirement of regularity at $r = 0$ in order to infer the value of $B(r)$ as $r \to 0$.

4. What should the behaviour of G^a_{0i} at $r\to\infty$ be in order for the energy (2) of the dyon to be finite? Show that

$$E^a_i := G^a_{0i} = -n_a\, n_i\, \frac{\partial B}{\partial r}, \tag{6}$$

and compute the gauge invariant electric field $\mathcal{E}_i = \frac{1}{v} E^a_i \phi^a_{vac}$ of the dyon.

5. Assume that the electric charge of the dyon equals Q_E. Show that the asymptotic behaviour for $B(r)$ at $r \to \infty$, consistent with this assumption and with the finiteness of the energy E, is

$$B = \frac{Q_E}{r} + C(Q_E), \tag{7}$$

where C is a constant that depends on the electric dyon charge Q_E.

6. Insert the Ansatz for the fields, Eqs. (3) and (5), into the energy functional (2) and find the field equations for the radial functions $B(r), F(r)$ and $H(r)$, by minimizing the energy functional.

7. Show that the asymptotic behaviour (7) for $B(r)$, with the boundary conditions at $r \to 0$ derived above, satisfies the equations in the limit of large r.