Advanced Quantum Field Theory

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Exercise Sheet 6

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Problem 1 - Solution of the Bogomol'nyi equation for monopoles

We have seen in the lectures that magnetic monopole solutions can arise in the Georgi-Glashow (GG) model. The Lagrangian of the model is

$$\mathcal{L} = -\frac{1}{4\,g^2} G^a_{\mu\nu}\, G^{\mu\nu,a} + \frac{1}{2}(\mathcal{D}_\mu \phi^a)(\mathcal{D}^\mu \phi^a) - \lambda\,(\phi^a\phi^a - v^2)^2. \tag{1}$$

The energy functional following from this Lagrangian in the limit $\lambda \to 0$ reads

$$E = \int d^3x \left[\frac{1}{2g^2} B^a_i B^a_i + \frac{1}{2}(\mathcal{D}_i \phi^a)(\mathcal{D}_i \phi^a) \right], \tag{2}$$

where the magnetic field is defined as

$$B^a_i = -\frac{1}{2}\epsilon_{ijk} G^a_{jk}\,,$$

and of course

$$G^a_{jk} = \partial_j A^a_k - \partial_k A^a_j + \epsilon^{abc} A^b_j A^c_k\,.$$

1. Show that the energy functional can be rewritten as

$$E = v\,Q_M + \int d^3x \left[\frac{1}{2} \left(\frac{1}{g} B^a_i - \mathcal{D}_i \phi^a \right) \left(\frac{1}{g} B^a_i - \mathcal{D}_i \phi^a \right) \right], \tag{3}$$

where the magnetic charge Q_M in a sphere of radius R is defined as the flux of the magnetic field through the surface of the sphere $\mathcal{S}_R$

$$Q_M = \frac{1}{g\,v} \int_{\mathcal{S}_R} d^2 S_i\, B^a_i\, \phi^a\,.$$

2. Minimizing the energy functional is equivalent to imposing the Bogomol'nyi equation

$$\frac{1}{g} B^a_i - \mathcal{D}_i \phi^a = 0\,. \tag{4}$$

By solving this set of *nine* first-order differential equations we can find the field configurations for the magnetic monopole solution. You have seen in class that the proper *Ansatz* for the fields is

$$\phi^a = v\, n^a\, H(r)\,, \qquad A^a_i = \epsilon^{aij} \frac{1}{r} n^j F(r)\,, \tag{5}$$

where $n^i = x^i/r$ and $H(r)$ and $F(r)$ are two scalar functions of r only. Note that we have written the *nine* different functions ϕ^a and the A^a_i in terms of *two* scalar functions only! It is very non trivial that this Ansatz is enough to solve the nine equations (4). Show by explicit calculation that with the Ansatz above we have

$$\begin{aligned} B^a_i &= \left(\delta^{ai} - n^a n^i\right) \frac{1}{r} F' + n^a n^i \frac{1}{r^2} \left(2F - F^2\right) \\ \mathcal{D}_i \phi^a &= v \left[\left(\delta^{ai} - n^a n^i\right) \frac{1}{r} H(1-F) + n^a n^i H' \right], \end{aligned} \tag{6}$$

with $H' = dH(r)/dr$, $F' = dF(r)/dr$.

3. Using the result above show that the Ansatz goes through and Eqs. (4) are equivalent to the system of two differential equations

$$\frac{dF}{d\rho} = H(1-F)\,, \qquad \frac{dH}{d\rho} = \frac{1}{\rho^2}\left(2F - F^2\right)\,, \tag{7}$$

in terms of the new rescaled variable $\rho = g\,v\,r$.

4. You have seen in the lecture what the solution to equations (7) look like. Try and solve the equations explicitly imposing the correct boundary condition.

Problem 2 - Invariance of monopole solution

Consider the monopole solution found for the fields ϕ^a and A_i^a. Prove explicitly that the monopole solution stays intact under the combined action $\vec{L} + \vec{T}$, where $\vec{L}$ and $\vec{T}$ denote respectively the generators of the spatial and $SU(2)$ color rotations.