Advanced Quantum Field Theory

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Exercise Sheet 5

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Problem 1 - The non-critical vortex in the small Higgs mass limit

In last week's exercise we derived the energy of the non-critical vortex for the case where the Higgs mass is of the same order of magnitude (but not exactly the same!) as the vector boson mass, i.e. $m_V \approx m_H$. The result was an energy of approximately $E \approx m_V^2/e^2$. On the other hand, during the lecture the energy of the vortex was shown to be $E \simeq 2\pi v^2 \log(m_H/m_V)$ in the case of $m_H \gg m_V$. In this exercise we will derive the energy of the vortex for the remaining case of $m_H \ll m_V$ (this case was first considered in [1]). The energy functional of the vortex is

$$E[\vec{A},\phi] = \int d^2x \left[\frac{1}{4\,e^2} F_{ij}F_{ij} + |\mathfrak{D}_i \phi|^2 + U(\phi) \right], \quad U(\phi) = \lambda \left(|\phi|^2 - v^2 \right)^2. \tag{1}$$

We take the same Ansatz for the fields as before

$$\phi(r) = v\,\eta(r)\,e^{i\,\theta}\,, \qquad A_i(x) = -\frac{1}{n_e}\epsilon_{ij}\frac{x_j}{r^2}\left(1 - f(r)\right), \tag{2}$$

where the boundary conditions are

$$\eta(0) = 0,\ \eta(\infty) = 1, \qquad f(0) = 1,\ f(\infty) = 0. \tag{3}$$

1. Show that the energy of the vortex as a function of η and f equals

$$E[f,\eta] = 2\pi \int dr \left\{ \frac{1}{2n_e^2 e^2}\frac{f'^2}{r} + r\,v^2\,\eta'^2 + v^2\,\frac{f^2}{r}\eta^2 + \lambda\,v^4\,(\eta^2-1)^2\,r \right\}. \tag{4}$$

2. Derive again the equations that η and f need to satisfy in order for the energy functional (4) to be minimized. The equations are

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d\,f}{d\,r}\right) - 2\,n_e^2\,e^2\,v^2\,\frac{\eta^2}{r}\,f = 0 \tag{5}$$

$$-\frac{d}{dr}\left(r\frac{d\,\eta}{d\,r}\right) + 2\lambda v^2\,r\,\eta\left(\eta^2-1\right) + \frac{\eta}{r}\,f^2 = 0. \tag{6}$$

3. In order to solve the above equations, we will assume a first order approximation that the electromagnetic field of the vortex is confined to its core of radius R_V (which we derive later), such that $A_i = 0$ at $r \geq R_V$ and $\eta = 0$ at $r \leq R_V$. At the end of the exercise we will improve upon this first order approximation. Demonstrate that in this approximation, the field f which solves (5) equals

$$f^{(0)} = \begin{cases} 1 - \frac{r^2}{R_V^2}, & r \leq R_V \\ 0, & r > R_V \end{cases} \tag{7}$$

4. Assuming that the non-linear term in $\xi := 1 - \eta$ can be neglected and also $m_H R_V \ll 1$ (we confirm this later below), show that the field η which solves (6) equals

$$\eta^{(0)} = \begin{cases} 0, & r \leq R_V \\ 1 - \frac{K_0(m_H r)}{\log(2/(m_H R_V))}, & r \gtrsim R_V \end{cases} \tag{8}$$

where the function $K_0(x)$ is a modified Bessel function of the second kind that satisfies (refer to [2]) the differential equation $x^2 \partial_x^2 K_0(x) + x \partial_x K_0(x) - x^2 K_0(x) = 0$ and behaves as[1] $K_0(x) \underset{x \to 0}{\simeq} \log(2/x) - \gamma$.

5. Substitute the first order solutions (7) and (8) in the energy functional (4) and show, again under the assumption $m_H R_V \ll 1$, that the energy of the vortex at first approximation equals

$$E_0 = 2\pi \left\{ \frac{1}{e^2 n_e^2 R_V^2} + \frac{v^2}{\log(2/(m_H R_V))} \left[1 + \mathcal{O}\left(\frac{1}{\log(1/(m_H R_V))} \right) \right] \right\}. \tag{9}$$

6. By minimizing the energy with respect to R_V, under the assumption that $m_H \ll m_V$, show that the leading logarithmic approximation is

$$R_V^2 \simeq \frac{4}{m_V^2} \log^2(m_V/m_H). \tag{10}$$

The above behaviour is consistent with our previous assumption that $m_H R_V \ll 1$. Furthermore, the second term in the energy functional is the main contribution to the energy from the field η. The last term in $E[f, \eta]$ is in fact only important for regularizing (cf. [1]) the behaviour of η at $r \to \infty$ and therefore we were justified in dropping the non-linear term when we derived $\eta^{(0)}$ above.

7. Show that the energy of the vortex to first order equals

$$E_0 = \frac{2\pi v^2}{\log(m_V/m_H)}. \tag{11}$$

8. Let us now consider small corrections to our first order approximations (7) and (8). By taking into account the first order (7) for f, together with the limiting behaviour for η at $r \simeq R_V$ as follows from (8), show that inside the core $r \leq R_V$ the field η behaves as

$$\eta \sim \frac{1}{\log(2/(m_H R_V))} \frac{r}{R_V} (1 + \mathcal{O}(r/R_V)), \quad r \leq r_V. \tag{12}$$

9. By using the improved approximation of η as found above, show that inside the core $r \leq R_V$ the field f equals

$$f = 1 - \frac{r^2}{R_V^2} + \mathcal{O}\left(\frac{r^4}{R_V^4} \right), \quad r \leq r_V. \tag{13}$$

10. Argue that in the logarithmic approximation, the above corrections to the fields η and f inside the core $r \leq R_V$ lead to the same energy for the vortex as given in (11).

References

[1] A. Yung, Nucl. Phys. B **562** (1999) 191 [hep-th/9906243].

[2] http://mathworld.wolfram.com/ModifiedBesselDifferentialEquation.html

[1] γ is Euler's constant