Advanced Quantum Field Theory

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Exercise Sheet 4

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Problem 1 - The asymptotic vector potential

In class we have seen that in order for the energy of the Vortex solution to be finite the vector potential must behave asymptotically as

$$A_i \underbrace{\longrightarrow}_{r\to\infty} -\frac{n}{n_e}\frac{\epsilon_{ij}\,x_j}{r^2} \tag{1}$$

where ϵ_{ij} is the Levi-Civita tensor in 2 dimensions, n_e is the electric charge of the field ϕ and n is the *winding number*. Prove that (1) is a pure gauge, i.e. that it can be removed by a gauge transformation.

Problem 2 - The equations of motion for the critical vortex

The critical vortex solution is defined by the condition that the masses of the gauge bosons m_V and the Higgs boson m_H are equal. In this case the equations of motion for the vortex can be solved and the energy of the stable configuration can be determined. In the case of a critical vortex this is equivalent to minimizing the *Bogomol'nyi completion* formula

$$T[\vec{A},\phi] = 2\,\pi\,n\,v^2 + \int d^2\vec{x}\,\left\{ \frac{1}{2}\left[\frac{B}{e} + n_e\,e\,(|\phi|^2 - v^2)\right]^2 + |\mathfrak{D}_1\phi + i\,\mathfrak{D}_2\phi|^2 \right\}. \tag{2}$$

1. Discuss why, to minimize the energy, we need to impose

$$\begin{cases} \frac{B}{e} + n_e\,e\,(|\phi|^2 - v^2) = 0 \\ \\ \mathfrak{D}_1\phi + i\,\mathfrak{D}_2\phi = 0\,. \end{cases} \tag{3}$$

   What is the mass of the stable critical vortex configuration?

2. Consider the case $n = 1$ and put for simplicity $m_V = m_H = 1$. Prove that the Ansatz

$$\phi(r) = v\,\eta(r)\,e^{i\,\theta}\,, \qquad A_i(x) = -\frac{1}{n_e}\epsilon_{ij}\frac{x_j}{r^2}\,(1 - f(r)) \tag{4}$$

   where $\eta(r)$ and $f(r)$ are two scalar functions which depend only on the radial coordinate r and θ is the polar angle, goes through the equations and one gets

$$\begin{cases} -\frac{1}{\rho}\frac{d\,f}{d\,\rho} + \eta^2 - 1 = 0 \\ \\ \rho\,\frac{d\,\eta}{d\rho} - f\,\eta = 0\,, \end{cases} \tag{5}$$

   where we defined the rescaled radial coordinate $\rho = n_e\,e\,v\,r$.

3. What are the boundary conditions for the two fields $\eta(\rho)$ and $f(\rho)$ as $\rho \to \infty$ and $\rho \to 0$?

4. Use any computer algebra program of your choice (Mathematica, Maple, Matlab,...) in order to solve numerically the differential equations Eq. (5).

5. What is the asymptotic behaviour of $f(\rho)$ and $\eta(\rho)$ for $\rho \to \infty$?

1 Problem 3 - The non-critical Vortex

We consider now the general case of a non-critical vortex, i.e. $m_V \neq m_H$. In this case the energy cannot be written as (2), and one must minimize the original form of the Energy

$$E[\vec{A}, \phi] = \int d^2x \left[\frac{1}{4\,e^2} F_{ij} F_{ij} + |\mathfrak{D}_i \phi|^2 + U(\phi) \right] , \tag{6}$$

with

$$U(\phi) = \lambda \left(|\phi|^2 - v^2 \right)^2 .$$

1. Using the same Ansatz as for the critical vortex, Eq. (4), and following the same steps you can prove that the functions $f(r)$ and $\eta(r)$ satisfy now the equations

$$\begin{cases} \frac{d}{dr}\left(\frac{1}{r}\frac{d\,f}{d\,r}\right) - 2\,n_e^2\,e^2\,\frac{\eta^2}{r}\,f = 0 \\ -\frac{d}{dr}\left(r\frac{d\,\eta}{d\,r}\right) + 2\lambda v^2\,r\,\eta\left(\eta^2 - 1\right) + \frac{\eta}{r}\,f^2 = 0 \end{cases} \tag{7}$$

2. A solution to these equations is not known. We can nevertheless use scaling considerations in order to estimate the mass of the non-critical vortex. Go back to the Energy functional (6) and, assuming that $m_V/m_H \approx 1$ (i.e. they are *not* equal but they are of the *same order of magnitude*!), perform the rescaling

$$\phi(\vec{x}) = v\,\eta(\vec{y}\,) , \qquad A_i(\vec{x}) = m_V C_i(\vec{y}\,) \quad \text{with} \quad \vec{y} = m_V\,\vec{x}\,. \tag{8}$$

   Show that under this rescaling the three terms in the energy functional are of the same order for

$$\eta \approx 1 , \quad C_i \approx 1 , \quad y \approx 1$$

   and in particular that

$$\frac{1}{e^2} F_{ij}^2 \approx |\mathfrak{D}_i \phi|^2 \approx \lambda \left(|\phi|^2 - v^2 \right)^2 \approx \mathcal{O}\left(\left(m_V^2/e \right)^2 \right) . \tag{9}$$

3. Write down the energy functional in the new variables and argue why its minimum should be attained for

$$\eta \approx 1 , \quad C_i \approx 1$$

   such that the characteristic size of the soliton is $y \approx 1$.

4. Using these results you can estimate the mass of the soliton from the value of (6) at its minimum. Show that

$$M_{soliton} \approx \frac{m_V^2}{e^2} .$$