Advanced Quantum Field Theory

V: Prof. Kirill Melnikov,   Ü: Dr. Lorenzo Tancredi

Exercise Sheet 3

Due x.11.2015

Problem 1 - Fermionic zero-mode in an anti-kink background

During the lecture the quantization of a fermionic field in a kink scalar background field was discussed in $1+1-$dimensional space-time. The Lagrangian density of the model is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{4}g^2(\phi^2 - v^2)^2 + i\bar{\psi}\hat{\partial}\psi + \lambda\phi\bar{\psi}\psi. \tag{1}$$

The scalar field admits a static kink solution $\phi_K(x)$ and if $v^2 \ggg \lambda/g$, it can be thought of as a time-independent background when quantizing the fermionic field ψ. Furthermore, it was shown in the lecture that the equation of motion for the fermion field in the scalar kink background, $(i\hat{\partial}+\lambda\phi_K(x))\psi = 0$, admits a static zero-mode solution $\psi = \begin{pmatrix} 0 \\ \chi_0 \end{pmatrix}$ with $\chi_0 \sim \exp(-\lambda \int_0^x \phi_K(x)) \underset{|x|\to\infty}{\longrightarrow} \exp(-\lambda v|x|)$.

1. Recall how charge conjugation and parity transformation act on the fermionic field ψ in $3+1-$dimensional space-time. Generalize this to the case of $1+1-$dimensional space-time.
2. By explicit calculation, show that the zero-mode solution is invariant under charge conjugation.
3. Show how to explicitly find the zero-mode solution in the anti-kink background by applying the parity transformation to the zero-mode solution in the kink background.

Problem 2 - Domain wall fermion model

One of the ways to study quantum field theory beyond perturbation theory, is to discretize the space-time; such an approach is called *lattice field theory.* We would like to study lattice field theories that are as close to a continuum limit as possible. One of the important features of realistic field theories is the so-called chiral symmetry, which is a symmetry of “rotating” left- and right-fermion fields independently from each other. Chiral symmetry is an important feature of strong and weak interactions. For example, the left- and right-handed fermions in the Standard Model have different charges w.r.t. electroweak interactions and this causes parity violating effects. Spontaneous breaking of the chiral symmetry is used to explain a tiny mass of the pion.

Amusingly, there is a no-go theorem (Nielsen-Nimomiya theorem) that states that a lattice theory for chiral fermions can not be formulated in a self-consistent way. This was an unfortunate fact for lattice calculations since it created a serious obstacle in designing theories compatible with the real world. It turns out that there is a way to get around this theorem - it is known as the domain wall fermion model. This model produces a chiral fermion in four dimensions from a lattice theory of a massive interacting fermion in five dimensions. We will work out the basics of this model below.

Consider a theory described by an action in a five-dimensional Minkowski space

$$\mathcal{S} = \int d^4x ds \bar{\psi}(x,s)\Big[i\Gamma_M\partial^M - m(s)\Big]\psi(x,s) \tag{2}$$

where $M = 0, ..4$ is the five-dimensional index, Γ_M are the elements of the Clifford algebra in five dimensions, s is the position label along the fifth dimension and $m(s)$ is the "mass" of the fermion defined as

$$m(s) = \begin{cases} m_0, & s > 0 \\ -m_0, & s < 0 \end{cases} \tag{3}$$

1. Use four-dimensional γ-matrices (which satisfy the four-dimensional Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$) to construct a representation of the Clifford algebra in five dimensions by choosing appropriate Γ_M's.
2. Write down the five-dimensional equation of motion for the field ψ. Solve this equation assuming that motion along the fifth dimension and in the four-dimensional space factorizes. Choose the four-dimensional part of the solution by attempting to describe a massless (in the four-dimensional sense) Dirac particle with momentum p.
3. Show, that by choosing the solution to be the eigenstate of the four-dimensional matrix γ_5, you can find $\psi(x, s)$ which is normalizable in the fifth dimension. This is the fermion localized on the "defect" in the fifth dimension which looks like a perfect chiral fermion from the point of a four-dimensional observer.