# Advanced Quantum Field Theory

V: Prof. Kirill Melnikov,   Ü: Dr. Lorenzo Tancredi

## Exercise Sheet 2

Due 04.11.2015

## Problem 1 - One-loop renormalization of kink mass

We consider again the two-dimensional kink. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \tag{1}$$

with

$$V(\phi) = \frac{1}{2}\left[\mathcal{W}'(\phi)\right]^2 , \qquad \mathcal{W}(\phi) = \frac{g}{\sqrt{2}}\left(\frac{\phi^3}{3} - v^2\,\phi\right) .$$

As always writing $\phi = v + \chi$, after SSB, the field $\chi$ develops a mass $m = \sqrt{2}\,g\,v$.

1. Using a UV cutoff $M_{uv}$ to regularize the theory, prove that the one-loop relation between the renormalized $m_R$ and the bare mass $m$ parameter reads

$$m_R^2 = m^2 - \frac{3\,g^2}{2\,\pi}\,\ln\frac{M_{uv}^2}{m^2}.$$

## Problem 2 - Casimir effect

Consider a theory of a free massless scalar field $\phi$ confined between infinitely large plates that are parallel to the $(x, y)$ plane, and located respectively at $z = 0$ and $z = L$.

1. Quantize the field assuming that it vanishes at $z = 0$ and $z = L$ at all times. To this end, write the field $\phi(t, \vec{r})$ as

$$\phi(t, \vec{r}) = \sum_n \sqrt{\frac{2}{L}} \int \frac{d^2\vec{k}_\perp}{(2\pi)^2}\,\phi_n(\vec{k}_\perp)\,e^{i\vec{k}_\perp\cdot\vec{r}_\perp}\,\sin\left(\frac{\pi\,n\,z}{L}\right), \tag{2}$$

where $\vec{k}_\perp$ and $\vec{r}_\perp$ are vectors in the $(x, y)$ plane. Express as usual $\phi_n(\vec{k}_\perp)$ and the canonical momentum $\pi = \partial\phi_n(\vec{k}_\perp)/\partial t$, through creation and annihilation operators $a^\dagger_{k_\perp,n}\,, a_{q_\perp,n}$.

2. Show that the commutation relations between $\phi$ and $\pi$ are satisfied if the creation and annihilation operators fulfil

$$\left[a_{k_\perp,n_1}\,, a^\dagger_{q_\perp,n_2}\right] = \delta_{n_1 n_2}(2\pi)^2\,\delta^{(2)}(\vec{k}_\perp - \vec{q}_\perp) \tag{3}$$

3. Show that the Hamiltonian reads

$$H = \sum_n \int \frac{d^2\vec{k}_\perp}{(2\pi)^2}\,\omega_{k_\perp,n}\left(a^\dagger_{k_\perp,n}\,a_{k_\perp,n} + \frac{A_\perp}{2}\right), \tag{4}$$

where $A_\perp$ is the area of the plates and

$$\omega_{k_\perp,n} = \sqrt{k_\perp^2 + \left(\frac{\pi\,n}{L}\right)^2}$$

4. Define the vacuum state $|0\rangle$ by the usual condition

$$a_{k_\perp,n}|0\rangle = 0$$

and show that, by acting on the vacuum state by creation operators, you produce states of definite energy. The last term in Eq. (4) is the vacuum energy.

5. Now in order to evaluate the vacuum energy, use dimensional regularization and modify the integration measure $d^2\vec{k}_\perp \to d^\nu\vec{k}_\perp$. Calculate the integral assuming that a value of $\nu$ exists such that the integral exists. Is it the case? You may make use of the following results

$$\int_0^1 dx\, x^{a-1}\,(1-x)^{b-1} = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,, \qquad \sum_{n=1}^{\infty}\frac{1}{n^\nu} = \zeta(\nu)\,, \tag{5}$$

where $\Gamma(x)$ is Euler Gamma function and $\zeta(x)$ is Riemann zeta function.

6. Show that after having explicitly evaluated the integral as a function of $\nu$, the limit $\nu \to 2$ can be safely taken giving for the Casimir energy

$$\frac{\mathcal{E}}{A_\perp} = -\frac{\pi^2}{1440\,L^3}\,. \tag{6}$$

7. The Casimir force can be obtained differentiating (6) with respect to $L$. Calculate the pressure between the two plates and compare it to the solar light pressure on Earth.