Advanced Quantum Field Theory

V: Prof. Kirill Melnikov,   Ü: Dr. Lorenzo Tancredi

Exercise Sheet 13

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Problem 1 - Finite-spacing corrections to the scalar propagator

We have seen in the lecture that it is straightforward to put a scalar field on a lattice. If the lattice spacing is a and the number of sites is N, the action for the field reads

$$S = \frac{a^4}{N^4} \sum_q \left\{ \frac{1}{2a^2} \sum_\mu (2 - 2\cos \mu \cdot q) \, |\phi_q|^2 + \frac{m^2}{2} \, |\phi_q|^2 \right\} . \tag{1}$$

From this action we can read off the propagator in position space

$$\langle \phi_{x_n} \phi_{x_m} \rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4 q}{(2\pi)^4} \frac{\mathrm{e}^{i\, q\, (x_m - x_n)}}{m^2 + \frac{2}{a^2} \sum_\mu \left(1 - \cos\left(a\, q \cdot \mu\right)\right)} \, , \tag{2}$$

In class we have seen that by expanding in a for $a \approx 0$ we recover at zeroth order the usual dispersion relation for massive particles in Euclidean space

$$\sum_\mu {q_\mu}^2 + m^2 = 0 \qquad \to \qquad \sqrt{-q_0^2} = E_0 = \sqrt{\vec{q}^{\,2} + m^2} \, , \tag{3}$$

where q_0 is the euclidean energy $q_0 = i\, E_0$.

1. Starting from (2), compute the first non-vanishing corrections in the lattice spacing a, to the dispersion relation (3).
2. Are there any continuous symmetries violated by these corrections?

Problem 2 - Finite-spacing corrections to gluon action in QCD

In class we studied in detail how to put gauge fields on the lattice by taking a product of Wilson lines, or links, that form a plaquette and tracing. In this way we get a gauge invariant contribution to the action. The Wilson lines are defined as

$$U_{x,\mu} = \mathrm{P} \exp \left(i\, g \int_0^a dt \, \hat{A}_\mu(x + \mu\, t) \right) \tag{4}$$

and the free lattice gauge action can be build starting from the plaquette operator

$$U^P_{\mu\nu} = \mathrm{Tr} \left(U_{x,\mu} \, U_{x+\mu,nu} \, U_{x+\mu+\nu,-\mu} \, U_{x+\nu,-\nu} \right) . \tag{5}$$

In the Abelian case the action is

$$S^{AB} = \frac{1}{g^2} \sum_{sites} \sum_{\mu<\nu} \mathrm{Re} \left(1 - U^P_{\mu\nu} \right) \tag{6}$$

while in the non-abelian case, for example for $SU(N)$, it reads

$$S^{SU(N)} = \frac{2\,N}{g^2} \sum_{sites} \sum_{\mu<\nu} \mathrm{Re} \left(1 - \frac{1}{2N} U^P_{\mu\nu} \right) \tag{7}$$

1. In class we showed that at order zero in a the abelian action (6) reduces to

$$S^{AB} \approx \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(a^2)\,. \tag{8}$$

   Repeat computation in the non-abelian case, i.e. starting from (7).

2. Compute the first corrections to Eq. (8), i.e. the term proportional to a^2.

3. The $\mathcal{O}(a^2)$ correction can be seen as due to higher dimensional operators in a effective field theory approach. Are the new interactions renormalizable? Do they violate any symmetries of the continuum theory?

4. Suppose that our Universe were really defined on a lattice and that the space-time were therefore intrinsically *not continuous*. Can you imagine any physical process which would receive corrections induced by the new higher dimensional operators above, and would allow us to discover the its discretized nature?