Advanced Quantum Field Theory

V: Prof. Kirill Melnikov,   Ü: Dr. Lorenzo Tancredi

Exercise Sheet 12

Due 03.02.2016

Problem 1 - Split regularization

In order to study the anomaly in 2-dimensions one needs to introduce a UV regulator, similar to a UV cut-off, which at the same time preserves the gauge invariance of the theory. This can be achieved using the so called *Schwinger splitting*, or split regularization. Instead of considering the original currents

$$j^\mu(t,x) = \bar{\psi}(t,x)\,\gamma^\mu\,\psi(t,x)\,, \qquad j_5^\mu(t,x) = \bar{\psi}(t,x)\,\gamma^\mu\gamma^5\,\psi(t,x)\,, \tag{1}$$

we consider the following regularized currents

$$\begin{aligned} j_\epsilon^\mu(t,x) &= \bar{\psi}(t,x+\epsilon)\,\gamma^\mu\,\psi(t,x)\exp\left(i\int_x^{x+\epsilon} A_1\,dx\right)\,, \\ j_{5,\epsilon}^\mu(t,x) &= \bar{\psi}(t,x+\epsilon)\,\gamma^\mu\gamma^5\,\psi(t,x)\exp\left(i\int_x^{x+\epsilon} A_1\,dx\right)\,, \end{aligned} \tag{2}$$

where A_1 is a gauge field. The parameter ϵ acts in this case as a regulator and the physically relevant results are recovered for $\epsilon \to 0$.

1. Show that the regularized currents (2) are indeed gauge invariant.
2. Prove that with these definitions the vector current is conserved.
3. Can you modify the definitions (2) and construct a similar split regularization that ensures conservation of the axial current instead of the vector current? Are the new split currents gauge invariant?

Problem 2 - The θ vacuum

In the lecture and in the previous problem we have seen how to introduce a UV regularization to treat the chiral anomaly in the Schwinger model. In this problem we will have a closer look at the vacuum of the theory, computing the energy of the Dirac sea.

We start writing down the fermion part of the Hamiltonian in the split-regularization as

$$H_\epsilon = \int_{-L/2}^{L/2} dx\,\psi(t,x+\epsilon)\,\sigma_3\,\left(i\frac{\partial}{\partial x} + A_1\right)\psi(t,x)\exp\left(i\int_x^{x+\epsilon} A_1\,dx\right)\,, \tag{3}$$

and the energy level of the "left-handed" and "right-handed" component of the Dirac sea look like

$$E_L = \sum_{k=0}^{\infty} E_{k(L)}\exp\big(i\,\epsilon\,E_{k(L)}\big)\,, \qquad E_R = \sum_{k=-1}^{-\infty} E_{k(R)}\exp\big(-i\,\epsilon\,E_{k(R)}\big)\,, \tag{4}$$

with $0 \le A_1 \le 2\pi/L$.

1. What is the physical meaning of Eqs. (4)? And how are the energy levels connected to the expression of the regularized "left-handed" and "right-handed" charges Q_L and Q_R seen in the lecture?

2. Show that, by expanding in ϵ, the total energy of the sea is

$$E_{sea} = E_L + E_R = \frac{L}{2\,\pi}\left(A_1^2 - \frac{\pi^2}{L^2}\right) + \text{ a constant independent of } A_1\,. \tag{5}$$

3. It follows from the previous item that the energy of the Dirac sea quadratically depends on A_1; as the result fluctuations of A_1 in the vacuum can be described by an effective Lagrangian

$$\mathcal{L} = \frac{L}{2e_0^2}(\partial_0 A_1)^2 - \frac{L}{2\pi}A_1^2\,. \tag{6}$$

   Use the analogy with the quantum oscillator and explain how to quantize the quantum system described by the above Lagrnagian. Determine the Hamiltonian and find the time dependence of the operator A_1. Find the ground state wave function of the quantum system described by the above Lagrangian $\Psi_0(A_1)$ and explain its physical meaning.

4. Our treatment of the Schwinger model was based on the assumption that we can find fermion energy levels considering the field A_1 to be time-independent. Given results obtained in the previous item and the energy levels of electrons in the Dirac sea, can you justify this approximation? You should look at the characteristic frequencies of the gauge field compared to those of the fermion sector.

5. We can now build up the vacuum wave function in the vicinity of $A_1 = 0$, in the form

$$\Psi_{vac} = \Psi_{ferm\ vac}\ \Psi_0(A_1)$$

   with

$$\Psi_{ferm.vac.} = \left(\prod_{k=0,1,2,..} |1_L, k\rangle\right)\left(\prod_{k=-1,-2,...} |0_L, k\rangle\right)\left(\prod_{k=-1,-2,..} |1_R, k\rangle\right)\left(\prod_{k=0,1,2,...} |0_R, k\rangle\right)\,. \tag{7}$$

   Show that this form of the wave function is not invariant under *large gauge transformations*, i.e. $A_1 \to A_1 + 2\,\pi\,k/L$ with $k = \pm 1, \pm 2\,...$

6. Show that for every $n = 0, \pm 1, \pm 2\,...$, such that $A_1 \approx 2\pi n/L$, the Hilbert space splits into distinct sectors corresponding to different structures of the fermion sea. For every sector, the vacuum wave functions of the fermion sea must be *restructured* as follows

$$\Psi^n_{ferm.vac.} = \left(\prod_{k=n}^{\infty} |1_L, k\rangle\right)\left(\prod_{k=n-1}^{-\infty} |0_L, k\rangle\right)\left(\prod_{k=n-1}^{-\infty} |1_R, k\rangle\right)\left(\prod_{k=n}^{\infty} |0_R, k\rangle\right)\,, \tag{8}$$

   and the vacuum wave function becomes, for every n,

$$\Psi_n = \Psi^n_{ferm.vac.}\Psi_0\left(A_1 - \frac{2\pi}{L}n\right)\,. \tag{9}$$

7. Show that the linear combination

$$\Psi_{\theta\ vac} = \sum_n e^{i\,n\,\theta}\Psi_n\,, \tag{10}$$

   independently of θ, is eigenfunction of the Hamiltonian with the lowest energy and is invariant under large gauge transformations $A_1 \to A_1 + 2\,\pi\,k/L$ (if we allow simultaneous renumbering of the energy levels). $\Psi_{\theta\ vac}$ is therefore a gauge invariant vacuum state. The parameter θ is called the *vacuum angle*.

8. What is the role of the new parameter θ? Is it an observable parameter? Which term should we add to the Lagrangian of the Schwinger model, in order to *imitate* the presence of the vacuum angle θ?