Advanced Quantum Field Theory

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Exercise Sheet 11

Due 27.01.2016

Problem 1 - Chiral Anomalies in Quantum Field Theory

Consider a fermionic theory described by the action

$$\mathcal{S} = \int d^4x \mathcal{L}(\psi, \partial\psi), \quad \mathcal{L} = i\,\bar{\psi}\gamma^\mu\partial_\mu\psi. \tag{1}$$

The above action is invariant under two classical symmetry transformations

$$\text{Vector} : \psi \to e^{i\alpha}\psi \qquad\qquad \text{Axial} : \psi \to e^{i\alpha\gamma_5}\psi \tag{2}$$

The matrix $\gamma_5 = i\,\gamma_0\gamma_1\gamma_2\gamma_3$ anticommutes with the Dirac matrices γ_μ. In this exercise we will show that the two symmetries are not simultaneously observed in the quantum theory. When a classical symmetry is not conserved under quantization one speaks of an *anomaly*. The anomaly of the fermionic quantum theory corresponding to the action (1) is called in the literature a *chiral anomaly*.

1. Show that the lagrangian (1) is invariant under the above two symmetries and their corresponding currents are conserved

$$\text{Vector current} : V_\mu = \bar{\psi}\gamma_\mu\psi \qquad\qquad \text{Axial current} : A_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi \tag{3}$$

2. Let us now calculate a correlator of three currents, namely

$$\left\langle 0 \left| T(A^\lambda(0)V^\mu(x_1)V^\nu(x_2)) \right| 0 \right\rangle = \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\,\Delta^{\lambda\mu\nu}(k_1,k_2)\,e^{ik_1x_1}\,e^{ik_2x_2}. \tag{4}$$

   Write the two connected diagrams that contribute to the correlator $\Delta^{\lambda\mu\nu}$ and show that

$$\Delta^{\lambda\mu\nu}(k_1,k_2) = (-1)i^3\int\frac{d^4l}{(2\pi)^4}\,\mathrm{Tr}\left\{\gamma^\lambda\gamma_5\frac{1}{\hat{l}-\hat{q}}\gamma^\nu\frac{1}{\hat{l}-\hat{k}_1}\gamma^\mu\frac{1}{\hat{l}} + \gamma^\lambda\gamma_5\frac{1}{\hat{l}-\hat{q}}\gamma^\mu\frac{1}{\hat{l}-\hat{k}_2}\gamma^\nu\frac{1}{\hat{l}}\right\}, \tag{5}$$

   where Tr denotes the trace, $\hat{k} := \gamma_\mu k^\mu$ and $q = k_1 + k_2$.

In what follows we will compute $k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2)$ and $q_\lambda\Delta^{\lambda\mu\nu}(k_1,k_2)$ and show that they cannot be *both* zero, which corresponds to either $\partial_\mu V^\mu$ or $\partial_\lambda A^\lambda$ being nonzero. As will be seen below, we will need to compute integrals of the following form

$$\Delta(a) = \int\frac{d^4l}{(2\pi)^4}\left(F(l_\mu + a_\mu) - F(l_\mu)\right). \tag{6}$$

Ordinarily an integral of the above form would be identical to zero for all a, by just shifting the variable $l \to l-a$ in the first term. However, whenever the integral over the function F diverges linearly, the integral $\Delta(a)$ is in fact not zero and depends on the value of the shift a. Henceforth we assume that $F(l) \sim \frac{1}{l^3}$ as $l \to \infty$.

4. Regularize the integral in (6) by multiplying the integrand by $\frac{\Lambda^2}{l^2+\Lambda^2}$ and let the integration be performed inside the 4-dimensional ball $B_4(\Lambda)$ with radius Λ. As long as $F(l) \sim \frac{1}{l^3}$, the integral will now converge as $\Lambda \to \infty$. Show that the result is

$$\Delta(a) = \lim_{\Lambda\to\infty} \int_{B_4(\Lambda)} \frac{d^4 l}{(2\pi)^4} \Big[F(l_\mu + a_\mu) - F(l_\mu)\Big] \frac{\Lambda^2}{\Lambda^2 + l^2} = \int \frac{d\Omega_4}{(2\pi)^4}\, l^2 (l\cdot a) F(l)|_{l\to\infty}, \tag{7}$$

where the last integration is over the angles parametrizing the 3-sphere.

5. Show that $k_{1\mu}\Delta^{\lambda\mu\nu}(k_1, k_2)$ can be written as

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1, k_2) = (-1)i^3 \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma^\lambda \gamma_5 \frac{1}{(\hat{l} - \hat{k}_1) - \hat{k}_2} \gamma^\nu \frac{1}{\hat{l} - \hat{k}_1} - \gamma^\lambda \gamma_5 \frac{1}{\hat{l} - \hat{k}_2} \gamma^\nu \frac{1}{\hat{l}} \right\}. \tag{8}$$

and argue that both terms under the integral sign converge as $\frac{1}{l^3}$ when $l \to \infty$. Therefore the result (7) is applicable.

6. Compute

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{8\pi^2}\epsilon^{\tau\nu\sigma\lambda} k_{2\tau} k_{1\sigma}, \tag{9}$$

where $\epsilon^{\tau\nu\sigma\lambda}$ denotes the 4-dimensional Levi-Civita symbol.

We have found that $k_{1\mu}\Delta^{\lambda\mu\nu}(k_1, k_2) \neq 0$ which is inconsistent with conservation of the vector current (3). Conservation of the vector current implies conservation of an electric charge, which we know to be conserved in nature. However, our findings imply that the Feynman integrals are not uniquely defined to begin with, since a shift in the internal loop momenta changes the final result. To take this ambiguity into account, let us parametrize the shift by considering the correlator

$$\begin{aligned} \Delta^{\lambda\mu\nu}(k_1, k_2, \{a_1, a_2\}) \;=\; & (-1)i^3 \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma^\lambda\gamma_5 \frac{1}{\hat{l} + \hat{a}_1 - \hat{q}} \gamma^\nu \frac{1}{\hat{l} + \hat{a}_1 - \hat{k}_1} \gamma^\mu \frac{1}{\hat{l} + \hat{a}_1} \right. \\ & \left. + \gamma^\lambda \gamma_5 \frac{1}{\hat{l} + \hat{a}_2 - \hat{q}} \gamma^\mu \frac{1}{\hat{l} + \hat{a}_2 - \hat{k}_2} \gamma^\nu \frac{1}{\hat{l} + \hat{a}_2} \right\}. \end{aligned} \tag{10}$$

7. Show that

$$\Delta^{\lambda\mu\nu}(k_1, k_2, \{a_1, a_2\}) = \Delta^{\lambda\mu\nu}(k_1, k_2, \{0, 0\}) + \frac{i}{8\pi^2}\epsilon^{\lambda\mu\nu\sigma}(a_1 - a_2)_\sigma. \tag{11}$$

8. Show that the conservation of the vector current implies that we must choose $(a_1 - a_2)_\sigma = (k_1 - k_2)_\sigma$ and therefore a *proper definition* of the vector conserving correlator is

$$\Delta^{\lambda\mu\nu}(k_1, k_2) := \Delta^{\lambda\mu\nu}(k_1, k_2, \{0, 0\}) + \frac{i}{8\pi^2}\epsilon^{\lambda\mu\nu\sigma}(k_1 - k_2)_\sigma. \tag{12}$$

9. Compute now

$$q_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{2\pi^2}\epsilon^{\mu\nu\lambda\sigma} k_{1\lambda} k_{2\sigma}. \tag{13}$$

The above result implies that in quantum theory the axial current is *not* conserved if the vector current is conserved and vice versa, even though in the classical theory both are. If the theory is gauged as $\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\,\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi$, it can be shown that

$$\partial_\mu A^\mu_{\text{axial}} = \partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) = \frac{e^2}{(4\pi)^2}\epsilon^{\mu\nu\alpha\sigma}F_{\mu\nu}F_{\alpha\sigma}. \tag{14}$$

10. Argue that in a gauged chiral theory $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_L(i\hat{D})\psi_L$, the gauge current satisfies $\partial_\mu(\bar{\psi}_L(i\gamma^\mu)\psi_L) = \frac{e^2}{2(4\pi)^2}\epsilon^{\mu\nu\alpha\sigma}F_{\mu\nu}F_{\alpha\sigma}$ and is therefore not conserved. As a result of this, can you think of any reasons why such a theory would pose difficulties? An important example of a theory where left and right handed fermions are gauged differently is the Standard Model, but it can be shown that the chiral anomalies cancel.