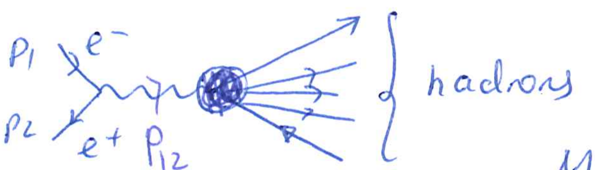
Lect. 9 | Operator product analysis of e^+e^- annihilation | -1-

I would like to discuss the connection between <u>hard</u> processes in QCD and <u>perturbative QCD</u> in some detail. The heuristic argument is that hard processes occur at short distances, so that the QCD coupling is small and the perturbative QCD works...
But - what is the accuracy of this statement? In other words - can we compute <u>corrections to predictions of perturbative QCD?</u>

To answer this question, consider the simplest process $e^+e^- \to$ hadrons at large center of mass energy $E_{CM} = \sqrt{s}$



The matrix element is written as

$$M_H = \frac{ie^2}{P_{12}^2} \bar{v}_{p2} \gamma^\mu u_{p1} \langle H | J_\mu^{em} | 0 \rangle ,$$

where $J_\mu^{em} = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s \equiv \sum_q Q_q \bar{q} \gamma_\mu q$
and $|H\rangle$ is a definite hadronic state.

In principle, if we are interested in a contribution of the final state H to $\sigma_{e^+e^- \to H_q}$, we should write $\langle H | J_\mu | 0 \rangle$ in terms of <u>invariant form-factors</u> (e.g. $|H\rangle = |\pi^+\pi^-\rangle$
$\langle \pi^+\pi^- | J_\mu | 0 \rangle = i F_\pi(p^2) (p_\mu^+ - p_\mu^-)$) and compute their contributions to cross-sections.

However, if we are interested ~~the~~ in inclusive cross-section, we need to sum over H. To this end, we write (massless leptons)

$$\sigma = \underbrace{\frac{1}{4}}_{\text{spin } e^+e^-} \frac{1}{4J} \sum_{\text{spins}} |M|^2 \, dLiPS = \frac{2\pi^2\alpha^2}{S^3} \mathrm{Tr}(\hat{p}_1 \gamma_\mu \hat{p}_2 \gamma_\nu) \times$$

$$\times \sum_H (2\pi)^4 \delta(P_{12} - p_H) \langle 0 | J_\nu^{em}(0) | H \rangle \langle H | J_\mu^{em}(0) | 0 \rangle$$

The set of hadronic states $|H\rangle$ is complete; we would like to use this ($\sum_H |H\rangle\langle H| = 1$) to get rid of the momentum conserving δ-function.

We write:

$$(2\pi)^4 \delta(P - p_H) = \int d^4x \, e^{i(P-p_H)_\mu x^\mu}$$

We now combine Heisenberg equation of motion for the current operator

$$J_\mu(x) = e^{i\hat{P}x} J_\mu(0) e^{-i\hat{P}x}$$

and the fact that $|H\rangle$ is the eigenstate of the momentum operator $e^{i\hat{P}x}|H\rangle = e^{ip_H \cdot x}|H\rangle$ to write

$$e^{-ip_H x} \langle 0 | J_\nu^{em}(0) | H \rangle = \langle 0 | e^{i\hat{P}x} J_\nu^{em}(0) e^{-i\hat{P}x} | H \rangle$$

$$\equiv \langle 0 | J_\nu^{em}(x) | H \rangle .$$

Then

$$\sigma = \frac{2\pi^2\alpha^2}{S^3} \mathrm{Tr}(\hat{p}_1 \gamma_\mu \hat{p}_2 \gamma_\nu) \int \sum_H e^{iPx} \langle 0 | J_\nu^{em}(x) | H \rangle \langle H | J_\mu^{em}(0) | 0 \rangle$$

$$\Rightarrow \boxed{\sigma = \frac{2\pi^2\alpha^2}{S^3} \mathrm{Tr}(\hat{p}_1 \gamma_\mu \hat{p}_2 \gamma_\nu) \int d^4x \, e^{iP_\mu x^\mu} \langle 0 | J_\nu^{em}(x) J_\mu^{em}(0) | 0 \rangle}$$

There is a useful way to rewrite the last integral to facilitate subsequent analysis. Consider the matrix element of the time-ordered product of two currents:

$$\Pi_{\mu\nu}(P) = i\int d^4x\, e^{iP_\mu x^\mu} \langle 0| T\, J_\mu(x) J_\nu(0) |0\rangle$$

Next, we move "backwards", i.e. write the T-product explicitly & introduce back hadronic states by ~~inserting~~ identity operators. We write

$$\Pi_{\mu\nu}(P) = i \sum_H \int d^4x\, e^{iPx} \Big\{ \theta(x_0)\, e^{-ip_H x} \langle 0| J_\mu(0)|H\rangle\langle H|J_\nu(0)|0\rangle + \theta(-x_0)\, e^{ip_H x} \langle 0| J_\nu(0)|H\rangle\langle H| J_\mu(0)|0\rangle \Big\}$$

Now, use

$$\int d^3\vec{x}\, e^{i\vec{q}\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{q})$$

$$\int_{-\infty}^{\infty} dx_0\, e^{iq_0x_0}\, \theta(x_0) = \frac{i}{q_0+i0}$$

$$\int_{-\infty}^{\infty} dx_0\, e^{iq_0x_0}\, \theta(-x_0) = \frac{-i}{q_0-i0}$$

Then: (working in $P=(\sqrt{s},\vec{0})$ frame)

$$\Pi_{\mu\nu}(P) = (2\pi)^3 \sum_H \Big\{ (-1)\, \delta^{(3)}(\vec{p_H}) \frac{\langle 0|J_\mu(0)|H\rangle\langle H|J_\nu(0)|0\rangle}{P_0 - p_H^0 + i0} + \delta^{(3)}(\vec{p_H}) \frac{\langle 0| J_\nu(0)|H\rangle\langle H|J_\mu(0)|0\rangle}{P_0 + p_H^0 - i0} \Big\}$$

Now, use $\dfrac{1}{a+i0} = \text{real} \mp i\pi\delta(a)$, to write

$$\Pi_{\mu\nu} = \text{real} + \frac{i}{2}\sum_H (2\pi)^4 \delta^{(4)}(P-p_H) \langle 0| J_\mu(0)|H\rangle\langle H|J_\nu(0)|0\rangle$$

$$= \text{real} + \frac{i}{2}\int d^4x\, e^{iP_\mu x^\mu} \langle 0| J^{em}_\mu(x) J^{em}_\nu(0)|0\rangle$$

Hence, we find

$$\int d^4x\, e^{iP_\mu x^\mu} \langle 0 | J_\mu(x) J_\nu(0) | 0 \rangle = 2\, \mathrm{Im} \left\{ i \int d^4x\, e^{iP_\mu x^\mu} \langle 0 | T J_\mu(x) J_\nu(0) | 0 \rangle \right\}$$

If we write $i \int d^4x\, e^{iP_\mu x^\mu} \langle 0 | T J_\mu(x) J_\nu(0) | 0 \rangle$

$$\equiv (-g_{\mu\nu} P^2 + P_\mu P_\nu)\, \Pi(P^2),$$ we find

$$\boxed{\sigma = \frac{16\pi^2\alpha^2}{s}\, \mathrm{Im}\, \Pi(s)}$$

Let us consider the formula for the Fourier ~~in~~ transform of the two-vector-currents correlator in the limit when $P^2 = -Q^2 < 0$. In this case the integration over x is limited to $x \sim 1/Q$. If $Q \to \infty$, $x \to 0$.

The product of two currents in this situation can be rewritten following the suggestion by K. Wilson:

$$\boxed{J_\mu(x)\, J_\nu(0) \underset{x \to 0}{\sim} \sum_i C^{(i)}_{\mu\nu}(x)\, \mathcal{O}^{(i)}(0)} \quad (*)$$

where $C^{(i)}_{\mu\nu}(x)$ is a function of x & $\mathcal{O}^{(i)}(0)$ is an operator composed of quantum fields and their derivatives. Eq. $(*)$ is known as the operator product expansion.

Eq. $(*)$ contains important information that follows from dimensional analysis. Quantum fields have positive mass dimensions, ~~as well as~~ derivatives $\frac{\partial}{\partial x^\mu}$ have positive mass dimensions as well.

Since operators $\mathcal{O}^{(i)}$ are assembled out of quantum fields and derivatives, the mass dimensions of those operators is also <u>positive</u>. However, since the mass dimension of $J_\mu(x)\, J_\nu(0)$ <u>is fixed</u>, the increasing mass dimension of $\mathcal{O}^{(i)}(0)$ must be compensated by the coefficient function $C^{(i)}_{\mu\nu}(x)$; this compensation means that more positive powers of x_α should appear in $C^{(i)}_{\mu\nu}(x)$. Since each power of x_α implies <u>suppression</u> $(x \to 0!)$, contributions of higher-dimensional operators to $(*)$ is ~~mass~~ suppressed.

Furthermore, we are interested in the vacuum expectation value of $J_\mu(x)\, J_\nu(0)$, which means that the relevant operators $\mathcal{O}^{(i)}(0)$ should be <u>Lorentz-scalars</u>. We will also choose them to be gauge-invariant.

Let us write the expansion of $J_\mu(x)\, J_\nu(0)$ including the leading terms

$$J_\mu(x)\, J_\nu(0) \approx C^{1}_{\mu\nu}(x)\, \hat{\mathbb{1}} + C^{\bar{q}q}_{\mu\nu}(x) \left(m \bar{q} q\right)(0) + C^{G^2}_{\mu\nu}(x) \left(G^a_{\mu\nu}\, G^{a\mu\nu}\right)(0) + \ldots$$

since $\dim_\mu(\psi) = \dim_\mu(\bar\psi) \equiv 3/2$
and $\dim_\mu(G^2) = 4$ and $\dim_\mu(x) = -1$,
we find $G^{(11)}_{\mu\nu}(x) \simeq \frac{1}{x^6}$, $G^{\psi\bar\psi}_{\mu\nu} \sim \frac{1}{x^2}$,
$G^{G^2}_{\mu\nu}(x) \sim \frac{1}{x^2}$. We can now calculate the Fourier transform in appropriate kinematics

$$i \int d^4x \, e^{ip_\mu x^\mu} J_\mu(x) J_\nu(0) \Big|_{p^2 = -Q^2 < 0} =$$

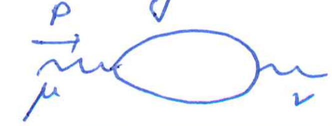
$$= i\left(-g_{\mu\nu} P^2 + P_\mu P_\nu\right) \Big\{ C^{1}(p^2) \cdot \hat{1} + C^{\psi\bar\psi}(p^2)\, m_q (\bar\psi\psi)(0) + C^{G^2}(p^2) \left(G^a_{\mu\nu} G^{a\mu\nu}\right)(0) + \ldots \Big\} \qquad (**)$$

In this case, the dimensional analysis implies

$$C^1(P^2) \sim (P^2)^0, \quad C^{\psi\bar\psi}(p^2) \sim (P^2)^{-2},$$

$$C^{G^2}(p^2) \sim (P^2)^{-2}.$$

Next, we would like to understand how to compute the coefficients that appear in the operator product expansion. Let us start with $C^1(p^2)$. To compute it, we can average Eq. $(**)$ over "perturbative vacuum" defined in such a way that $_{pert}\langle 0 | \mathcal{O}^{(i)}(0) | 0 \rangle_{pert} \equiv \phi$.

Then, $C^{(1)}(p^2)$ is given by a regular Feynman diagram . We find

$$\boxed{C^{(1)}(p^2) = -\frac{N_c}{12\pi^2} \sum_q Q_q^2 \ln(-p^2) + \ldots ,}$$

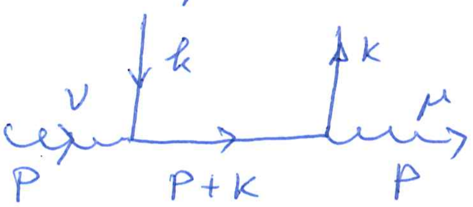
where ellipses stand for the uncalculated constant.

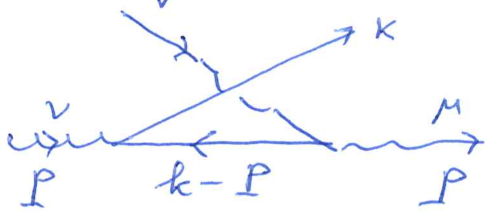
To illustrate how to compute other coefficients, we focus on $C^{\psi\bar\psi}(p^2)$ and consider the expectation value of $\int d^4x\, e^{i p_\mu x^\mu} T J_\mu(x) J_\nu(0)$ with respect to two single-quark states $|k\rangle$, described by the Dirac spinor u_k. We have

$$\langle k | \hat\Pi_{\mu\nu}(p) | k \rangle = \cancel{-Q_q^2 \bar u_k \left(\hat\gamma_\mu \frac{1}{\hat p + \hat k - m_q} \gamma_\mu \hat\gamma_\nu \frac{1}{\hat p - \hat k} \right.}$$

$$= -Q_q^2 \bar u_k \left[\hat\gamma_\mu \frac{1}{\hat p + \hat k - m} \hat\gamma_\nu + \hat\gamma_\nu \frac{1}{\hat k - \hat p - m_q} \hat\gamma_\mu \right] u_k , \quad (***)$$

where the contribution comes from the compton scattering diagrams





On the other hand, this result can be computed using the "right-hand-side" of the operator product expansion formula in Eq. (**).

We find:

$$\langle k|\,\hat{\Pi}_{\mu\nu}(P)\,|k\rangle = \left(-g_{\mu\nu}P^2 + P_\mu P_\nu\right) C^{\psi\bar\psi}(P^2) \times$$

$$\times \langle k|\, m_q(\bar\psi\psi)(0)\,|k\rangle =$$

$$= \left(-g_{\mu\nu}P^2 + P_\mu P_\nu\right) C^{\psi\bar\psi}(P^2)\, m_q(\bar u_k \cdot u_k).$$

Now, we go back to Eq. (***) and expand it in $k, m \ll P$. We find

$$\langle k|\,\hat{\Pi}_{\mu\nu}(P)\,|k\rangle = -Q_q^2\, \bar u_k \Big[\hat\gamma_\mu \frac{(\hat P + \hat k + m)}{(P+k)^2 - m^2} \hat\gamma_\nu$$

$$- \hat\gamma_\mu \frac{(\hat P - \hat k - m)}{(P-k)^2 - m^2} \gamma_\nu \Big] u_k; \quad \left[\underline{\text{Contract with } g^{\mu\nu}}\right]:$$

$$\langle k|\Pi^\mu_\mu|k\rangle = -Q_q^2\, \bar u_k \left[\frac{(-2(\hat P + \hat k) + 4m)}{P^2 + 2Pk} - \frac{(-2(\hat P - \hat k) - 4m)}{P^2 - 2P\cdot k} \right] u_k$$

$$= -Q_q^2\, \bar u_k \left[\frac{-2\hat P + 2m}{P^2 + 2P\cdot k} + \frac{2\hat P + 2m}{P^2 - 2P\cdot k} \right] u_k,$$

where we used equations of motion $\hat k u_k = m u_k$

~~thinks,~~ Next we write this as:

$$\langle k|\,\Pi^\mu_\mu(P)\,|k\rangle \approx -Q_q^2\, \bar u_k \left[\frac{2\hat P\, 4P\cdot k}{(P^2)^2} + \frac{4m}{P^2} \right] u_k$$

and average over <u>directions</u> of P (to project on a Lorentz scalar)

since $\langle 2\hat P\, 4P\cdot k\rangle_P = 2\hat k P^2$, we find

$$\langle k|\,\Pi^\mu_\mu(P)\,|k\rangle = -Q_q^2 (\bar u_k u_k) \frac{6m}{P^2}$$

Now, compare this with the OPE part:

$$g_{\mu\nu} \langle k | \hat{\Pi}_{\mu\nu}(p) | k \rangle = g^{\mu\nu} (-g_{\mu\nu} P^2 + P_\mu P_\nu)\, C^{\psi\bar{\psi}}(P^2) \times m_q (\bar{u}_k u_k) =$$

$$= -3P^2 C^{\psi\bar{\psi}}(P^2) \times m_q (\bar{u}_k u_k).$$

Hence, we find:

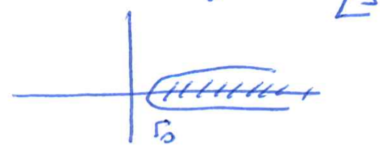
$$C^{\psi\bar{\psi}}(P^2) = \frac{2Q_q^2}{(P^2)^2}$$

Calculation of the coefficient C^{G^2} can be performed along similar lines. We find

$$\boxed{\begin{aligned} \Pi(P^2) = \langle 0 | \hat{\Pi}(P^2) | 0 \rangle &= -\frac{N_c}{12\pi^2} \sum_q Q_q^2 \ln(-P^2) \\ &+ \frac{2}{(P^2)^2} \sum_q Q_q^2 \langle m \bar{\psi}_q \psi_q \rangle + \\ &+ \frac{1}{12(P^2)^2} \sum Q_q^2 \left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^{a\mu\nu} \right\rangle + O(P^{-6}) \end{aligned}}$$

Note that this result is valid at $P^2 < 0$. On the other hand, the relation between the cross-section $\sigma_{e^+e^- \to hadrons}$ & $\Pi(P^2)$ exists for positive values of P^2. An important question is how the two regions can be connected.

To this end, we use the fact that the function $\Pi(s)$ is an analytic function of the variable s with the cut along positive real axis:



$$\Pi(Q^2) = \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds\ \mathrm{Im}\,\Pi(s)}{s - Q^2 - i0}$$

Since $\sigma(s) = \frac{16\pi^2\alpha^2}{s}\mathrm{Im}\,\Pi(s)$, we find

$$\Pi(Q^2) = \frac{1}{16\pi^3\alpha^2}\int_{s_0}^{\infty}\frac{ds \cdot s\,\sigma(s)}{s - Q^2 - i0} \qquad (****)$$

In principle, we may need to add subtraction terms but they will be not important for our purposes. Next, we take $Q^2 = -q^2 < 0$ and take $q^2 \to \infty$.

$$\Pi(-q^2) = -\frac{N_c}{12\pi^2}\sum_q Q_q^2 \ln(q^2) + O\left(\frac{1}{q^2}\right)$$

To obtain this logarithm from dispersion representation, we need to have $s\,\sigma(s) \longrightarrow$ const as $s \to \infty$.

To see this explicitly, write $s\sigma(s) = A$.

Then $$\int_{s_0}^{\infty}\frac{ds\ s\,\sigma(s)}{s + q^2} = A\int_{s_0}^{\Lambda^2}\frac{ds}{s+q^2} \longrightarrow -A\ln q^2$$

$$\Rightarrow \frac{N_c}{12\pi^2}\sum_q Q_q^2 \equiv \frac{A}{16\pi^3\alpha^2} \Rightarrow$$

$$A = \frac{4N_c\pi\alpha^2}{3}\sum_q Q_q^2 \Rightarrow \boxed{\sigma_{e^+e^-\to h}(s) \longrightarrow \frac{4N_c\pi\alpha^2\sum Q_q^2}{3s}}$$

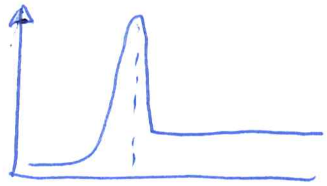
This is the high-energy asymptotic of the cross-section $e^+e^- \to$ hadrons (see h.w

As we see, we are able to ~~associate~~ relate the leading term in the OPE in an Euclidian region with high-energy behaviour of the $e^+e^- \to$ hadrons cross-section. ~~The~~ However, there are power-suppressed terms in the OPE that involve condensates. What kind of information do these terms contain that is relevant for $e^+e^- \to$ hadrons?

The answer is that the condensates give us some information about ~~vector mesons~~ non-perturbative - hadronic - states of QCD.

Indeed, if we look at $\sigma_{e^+e^- \to \text{hadrons}}$, we can display it, roughly, as

$s\sigma(s)$



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We approximate it like

$$\sigma(s) = \frac{A}{s} + B\,\delta(s - m_\rho^2)$$

The first term matches the leading term in the OPE.

The second gives the following contribution

$$\Delta\Pi(-q^2) \to \frac{1}{16\pi^3\alpha^2}\, B\, \frac{m_\rho^2}{m_\rho^2 + q^2} \simeq \frac{B}{16\pi^3\alpha^2}\left(\frac{m_\rho^2}{q^2} - \frac{m_\rho^4}{q^4}\right) + \ldots$$

So power corrections are related to non-perturbative contributions to cross-sections!

The idea of the sum rule method is to determine properties of resonances by equating power corrections to $\Pi(-q^2)$ with condensates, i.e. non-pert. matrix elements of local operators