Lect. 8.

Anomalies and the chiral Lagrangian

Consider a theory with massless fermions

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \, , \quad D_\mu = \partial_\mu + ie A_\mu .$$

We can split $\mathcal{L}$ into two parts by writing $\psi = \psi_L + \psi_R$, $\psi_{L,R} = \frac{(1 \pm \gamma_5)}{2} \psi$.

$$\mathcal{L} = \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad (*)$$

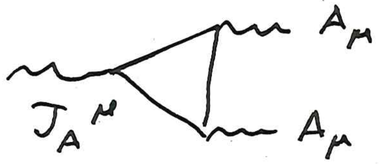
The Lagrangian Eq. (*) is invariant under $\psi_L \to e^{i\theta_L} \psi_L$, $\psi_R \to e^{i\theta_R} \psi_R$. This invariance implies the existence of two conserved currents $J_\mu^{R,L} = \psi_{R,L} \gamma_\mu \psi_{R,L}$.

$$\partial_\mu J_\mu^{R,L} \equiv 0 .$$

We can trade the left- and right currents for vector & axial currents

$$J_V^\mu = \bar{\psi} \gamma^\mu \psi \; ; \quad J_A^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi .$$

By computing triangle diagrams, we find that the conservation of the axial current and the vector current can not be maintained at the same time.

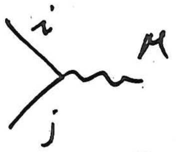


We choose to impose conservation of the vector current; then the equation for the divergence of the axial current becomes $\partial_\mu J_A^\mu = \frac{\alpha}{2\pi} F_{\mu\nu} \tilde{F}^{\mu\nu}$, which implies that the axial current is not conserved.

Suppose we want to generalize this result to ~~[illegible]~~ QCD with two flavors (u,d). The theory is invariant under the $SU(2)_L \otimes SU(2)_R$ transformations. The three axial currents are $j_A^{\mu,a} = \bar{\psi}\, i \frac{\tau^a}{2} \gamma_\mu \gamma_5 \psi$

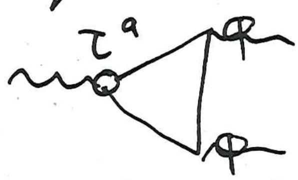
For an $SU(2)$ spinor $\binom{u}{d}$, the ~~current~~ electric charge operator is $\hat{Q} = \frac{1}{6}\hat{I} + \frac{1}{2}\hat{\tau}_3$.

The quark-photon-vertex is



$= -ie\hat{Q}_{ij}\hat{\gamma}^\mu$, so that the

~~come~~ anomaly equation ~~tegles~~ receives additional factors



$\sim Tr\left(\frac{\tau^a}{2}\hat{Q}\hat{Q}\right)$

$$\hat{Q}\hat{Q} = \frac{1}{36}\hat{I} + \frac{2}{12}\hat{\tau}_3 + \frac{1}{4}\tau_3^2 = \left(\frac{1}{36} + \frac{1}{4}\right)\hat{I} + \frac{1}{6}\hat{\tau}_3$$

Since $Tr(\tau^a \cdot \hat{I}) = 0$, only the second term $(\sim \frac{1}{6}\tau_3)$ works & we obtain

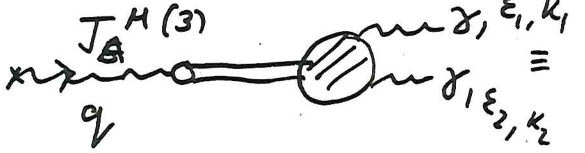
$$Tr\left(\frac{\tau^a}{2} Q Q\right) = \frac{1}{12}\, 2\delta^{a3} = \frac{1}{6}\delta^{a3}.$$

In addition, since quarks have 3 colors (N_c), the anomaly equation receives additional factor of $N_c = 3$. Hence, the anomaly equation in QCD becomes

$$\boxed{\partial_\mu J_A^{\mu,a} = \frac{\alpha N_c}{12\pi}\delta^{a3} F_{\mu\nu}\tilde{F}^{\mu\nu}.}$$

This equation allows us to compute pion (π^0) decay to two photons from first principles. Since pions are Goldstone bosons, their matrix elements of the axial current between vacuum and the pion state is not vanishing $\langle 0 | J_A^{\mu, a} | \pi^b \rangle = F_\pi \delta^{ab} \cdot p^\mu$

This implies that at low values of q^2, we should have



$$\equiv \frac{F_\pi q^\mu}{q^2} A_{\pi^0\to\gamma\gamma} \epsilon_{\mu\nu\alpha\beta} \varepsilon_1^\mu k_1^\nu \varepsilon_2^\alpha k_2^\beta .$$

+ terms without a pole.

Then
$$q_\mu J_{5\mu}^3 = \frac{4\alpha N_c}{12\pi} \epsilon_{\mu\nu\alpha\beta} \epsilon_1^\mu k_1^\nu \varepsilon_2^\alpha k_2^\beta = F_\pi A_{\pi^0\to\gamma\gamma} \cdot \varepsilon_1^\mu k_1^\nu \varepsilon_2^\alpha k_2^\beta$$
$+ \ldots ;$

The first ~~equation~~ step above follows from the anomaly equation. Disregarding non-pole terms, we find the amplitude for $\pi^0 \to \gamma\gamma$

$$\boxed{A_{\pi^0\to\gamma\gamma} \equiv \frac{\alpha N_c}{3\pi F_\pi}} \qquad (*)$$

We can now ask the following question: is anomaly ~~par~~ & decay $\pi^0 \to \gamma\gamma$ which as we saw is related to it, part of the chiral Lagrangian? The answer is no. Indeed, the ~~amplitude~~ Lagrangian

$$\mathcal{L}_{eff} = \frac{\alpha N_c}{12\pi F_\pi} \pi_0 F_{\mu\nu} \tilde{F}^{\mu\nu}$$

gives the $\pi^0 \to \gamma\gamma$ amplitude in Eq. (*) but ~~all~~ none of the terms in the chiral Lagrangian coupled to the external EM field seem to produce it. It was shown by Witten, Zumino and Wess that we need to introduce a new term to the chiral Lagrangian, to accommodate decay $\pi^0 \to \gamma\gamma$. (and similar effects) I'll try to explain this by looking at a much simplified example. (C. Hill, R. Hill, J. Harvey)

Consider a theory of a single massless quark and a simple lepton defined by the action:

$$S = \int d^4x \left(L_q + L_e \right)$$

$$L_q = \bar{q}_L (i\hat{\partial} + \hat{A}_L)\, q_L + \bar{q}_R (i\hat{\partial} + \hat{A}_R)\, q_R$$

$$L_e = \bar{\ell}_e (i\hat{\partial} - \hat{A}_L)\, \ell_e + \bar{\ell}_R (i\hat{\partial} - \hat{A}_R)\, q_R.$$

Here A_L and A_R are the two gauge fields. The theory is formally invariant under $U(1)_L \otimes U(1)_R$ gauge transformations.

The gauge transformations are defined as

$$q_L \to e^{i\epsilon_L} q_L, \quad q_R \to e^{i\epsilon_R} q_R,$$

$$\ell_L \to e^{-i\epsilon_L} \ell_e, \quad \ell_R \to e^{-i\epsilon_R} \ell_R,$$

$$\delta A^\mu_L \equiv \partial^\mu \epsilon_L, \quad \delta A^\mu_R \equiv \partial^\mu \epsilon_R.$$

~~On the other~~ hand, we know that

These transformations (global version) implies that the left- and right-currents for quarks & leptons are conserved. This, however, is not true ~~since~~ ~~the~~ because of the anomaly.

In particular, if $J_\mu^{q,(L,R)} = \bar{q}_{L,R}\gamma_\mu q_{L,R}$, then:

$$\partial^\mu J_\mu^{q,L} = -\frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\partial^\mu A_L^\nu\,\partial^\rho A_L^\sigma$$

$$\partial^\mu J_\mu^{q,R} = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\partial^\mu A_R^\nu\,\partial^\rho A_R^\sigma .$$

Similar equations exist for the lepton current $J_\mu^{\ell,(L,R)} = -\bar{\ell}_{L,R}\gamma_\mu \ell_{L,R}$:

$$\partial_\mu J_\mu^{\ell,(L,R)} = -\partial_\mu J_\mu^{q,(L,R)} .$$

Then $\partial_\mu\left(J_\mu^{q,(L,R)} + J_\mu^{\ell,(L,R)}\right) = \partial_\mu\left(\bar{q}_{L,R}\gamma^\mu q_{L,R} - \bar{\ell}_{L,R}\gamma_\mu \ell_{L,R}\right)$

$$\equiv \emptyset$$

Therefore, in our theory, full left and right currents are conserved but it is not the case for quark & lepton currents separately.

Now, let's imagine that our $U(1)_L \otimes U(1)_R$ breaks down to $U(1)_{V=L+R}$ and we get an associated Goldstone boson - a pion.

We would like to write an effective Lagrangian for this case assuming that quarks do not appear in the chiral Lagrangian.

The pion field (ϕ) transforms non-linearly under $\mathcal{U}(1)_L \otimes \mathcal{U}(1)_R$:

$$\phi \to \phi + \delta\phi \; ; \; \delta\phi \equiv F_\pi (\epsilon_L - \epsilon_R)$$

under $\mathcal{U}(1)_L \otimes \mathcal{U}(1)_R$, the change of the action is $\delta S = - \int \epsilon (\partial_\mu J^\mu)$. Hence

$$\delta S_q = \frac{1}{24\pi^2} \int (\epsilon_L \, dA_L \wedge dA_L - \epsilon_R \, dA_R \wedge dA_R) \, d^4x,$$

where $(dA)_L \wedge (dA)_L = \epsilon^{\mu\nu\alpha\beta} [\partial_\mu A_L^\nu][\partial_\alpha A_L^\beta]$

and $\delta S_L = - \delta S_q$, so that $\delta S = \delta S_L + \delta S_q \equiv \phi$.

If quarks are replaced by Goldstone bosons, we still have δS_L contribution to δS but the quark contribution is absent.

Nevertheless, if we write

$$S = S_L + \Gamma_{WZW}[\phi, A_L, A_R] + \Gamma_{ch}[\phi, A_L, A_R]$$

and require $\delta S = 0$ under $\mathcal{U}(1)_L \otimes \mathcal{U}(1)_R$, and assume that the standard chiral action $\Gamma_{ch}[\phi, A_L, A_R]$ is invariant, we find $\delta \Gamma_{WZW}[\phi, A_L, A_R] = - \delta S_L \Rightarrow$

end.—

$$\delta \Gamma_{WZW}[\phi, A_L, A_R] = \frac{1}{24\pi^2} \int [\epsilon_L (dA_L) \wedge (dA_L) - \epsilon_R (dA_R) \wedge (dA_R)]$$

Since $\delta\phi = F_\pi (\epsilon_L - \epsilon_R)$, we write

$$\Gamma_{WZW} = \frac{1}{24\pi^2} \int d^4x \, \frac{\phi}{F_\pi} \left((dA_L) \wedge (dA)_L + (dA_R) \wedge (dA_R) \right) + \Gamma^{(1)}(\phi, A_L, A_R)$$

where $\Gamma^{(1)}$ is unknown.

To compute the variation of Γ_{WZW}, note that $\delta\left((dA)_L \wedge (dA)_L\right) = d\delta A_L \wedge (dA)_L + (dA_L) \wedge d(\delta A_L) \equiv 0$, since $\delta A_L = \partial_\mu \epsilon_L \Rightarrow$

$$d(\delta A_L) \wedge (dA)_L = \epsilon_{\mu\nu\alpha\beta} (\partial_\mu \partial_\nu \epsilon_L) \partial_\alpha A_L^\beta = 0.$$

$$\Rightarrow \delta\Gamma_{WZW} = \frac{1}{24\pi^2} \int d^4x \, (\epsilon_L - \epsilon_R) \left(dA_L \wedge dA_L + dA_R \wedge dA_R \right) + \delta\Gamma^{(1)}(\phi, A_L, A_R) =$$

$$= \frac{1}{24\pi^2} \int d^4x \left[\epsilon_L \, dA_L \wedge dA_L + \epsilon_R \, dA_R \wedge dA_R - \epsilon_R \, dA_L \wedge dA_L - \epsilon_R \, dA_R \wedge dA_R \right] + \delta\Gamma^{(1)}(\phi, A_L, A_R).$$

Now, taking $\delta\Gamma_{WZW} \equiv -\delta S_k$, we find

$$0 = \delta\Gamma^{(1)}(\phi, A_L, A_R) + \frac{1}{24\pi^2} \int d^4x \left(\epsilon_L \, dA_R \wedge dA_R - \epsilon_R \, dA_L \wedge dA_L \right).$$

Now, write $\Gamma^{(1)}(\phi, A_L, A_R) = \frac{1}{24\pi^2 F} \int \phi \, dA_L \wedge dA_R + \Gamma^{(2)}$.

It follows

$$0 = \delta\Gamma^{(2)}(\phi, A_L, A_R) + \frac{1}{24\pi^2} \int \Big[(\epsilon_L - \epsilon_R)(dA_L \wedge dA_R) + \epsilon_L \, dA_R \wedge dA_R - \epsilon_R \, dA_L \wedge dA_L \Big]$$

$$\Rightarrow \delta\Gamma^{(2)}(\phi, A_L, A_R) = -\frac{1}{24\pi^2} \int d^4x \Big[\epsilon_L \, dA_L \wedge dA_R - \epsilon_R \, dA_L \wedge dA_R + \epsilon_L \, dA_R \wedge dA_R - \epsilon_R \, dA_L \wedge dA_L \Big].$$

We want to use the fact that $\delta A_L^\mu = \partial^\mu \epsilon_L$ and $\delta A_R^\mu = \partial^\mu \epsilon_R$. Therefore, we need to integrate by parts in each term to move one of the derivatives to "$\epsilon_{L,R}$".

The 4 terms are split into two pairs, depending on the number of left- and right fields that they contain. (counting $\epsilon_L \sim A_L$ and $\epsilon_R \sim A_R$). Then

$$Pair_1 = \varepsilon_L \, dA_L \wedge dA_R - \varepsilon_R \, dA_L \wedge dA_L$$

$$Pair_2 = \varepsilon_L \, dA_R \wedge dA_R - \varepsilon_R \, dA_L \wedge dA_R$$

To simplify these:

$$Pair_1 = \varepsilon_L \, dA_L \wedge dA_R - \varepsilon_R \, dA_L \wedge dA_L =$$

$$= - dA_L \wedge d\varepsilon_L \wedge A_R + d\varepsilon_R \wedge A_L \wedge dA_L$$

$$= - d\varepsilon_L \wedge A_R \wedge dA_L - A_L \wedge d\varepsilon_R \wedge dA_L$$

$$= - \delta A_L \wedge A_R \wedge dA_L - A_L \wedge \delta A_R \wedge dA_L =$$

$$= - \delta (A_L \wedge A_R \wedge dA_L), \text{ since } \wedge \delta(dA_L) = 0$$

$Pair_2$ is simplified in a similar way and we obtain

$$\delta \Gamma^{(2)} = \frac{1}{24\pi^2} \, \delta \int (A_L \wedge A_R \wedge dA_L + A_L \wedge A_R \wedge dA_R)$$

Now, if we put everything together, we find the low energy part of the action whose variation reproduces the anomalous behavior of the quark action:

$$\Gamma = \frac{1}{24\pi^2} \int \frac{\Phi}{F} (dA_L \wedge dA_L + dA_R \wedge dA_R + dA_L \wedge dA_R)$$
$$+ \frac{1}{24\pi^2} \int (A_L \wedge A_R \wedge dA_L + A_L \wedge A_R \wedge dA_R) .$$

This action predicts the decay of the Goldstone boson ϕ to two gauge bosons (the first term), as required by the anomaly. A generalization of this construction to the case of 3 flavor QCD where the $SU(3)_L \otimes SU(3)_R$ symmetry is spontaneously broken to $SU(3)_{L+R}$ is called the Wess-Zumino-Witten Lagrangian.

Let's review how this works.

First, we need the anomaly equation for the axial current. This is similar to the SU(2) case that we discussed; the only difference is that the electric charge operator changes. Taking the SU(3) "flavor" spinor to be $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$, we find $\hat{Q} = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$ and

$$\hat{Q}^2 = \frac{2}{9}\hat{1} + \frac{1}{3} T^{(3)}_{\beta} + \frac{1}{3\sqrt{3}} T^{(8)}, \text{ where}$$

$T^{(3,8)}$ etc. are the SU(3) generators.

Since $\partial_\mu J^a_{\mu,A} \sim Tr(T^a \hat{Q}^2)$, where $J^a_{\mu,A} = \bar{\psi} T^a \gamma^\mu \gamma_5 \psi$,

we find

$$\partial_\mu J^{(a),\mu}_A = \frac{N_c \alpha}{2\pi} F_{\mu\nu} \tilde{F}^{\mu\nu} \left(\frac{1}{6} \delta^{a3} + \frac{1}{6\sqrt{3}} \delta^{a8} \right) \quad (*)$$

The above equation gives us a change of the action under "axial" subgroup of $SU(3)_L \otimes SU(3)_R$ transformations.

We will need to construct a term in the Lagrangian that consists of

~~pi~~ Goldstone fields and the current that produces just the right variation of the action.

In reality, the story is more complex since there are more "anomalies" than indicated by Eq. (*) and there are constraints on anomalies that we impose (e.g. the vector currents $J_V^{\mu,a} = \bar{\psi}\gamma^\mu T^a \psi$ are <u>not anomalous</u>). So the construction proceeds as follows.

First, we consider our $SU(3)_L \otimes SU(3)_R$ coupled to left- and right- gauge fields. The gauge transformations are

$$\Sigma \to L \Sigma R^+ ; \quad \begin{cases} \hat{\ell}^\mu \to L \hat{\ell}^\mu L^+ - i L \partial^\mu L^+ \\ \hat{r}^\mu \to R \hat{r}^\mu R^+ - i R \partial^\mu R^+ \end{cases}$$

$L \in SU(3)_L$, $R \in SU(3)_R$

It is convenient to define vector $v^\mu = \frac{\ell^\mu + r^\mu}{2}$ and axial $a^\mu = \frac{\ell^\mu - r^\mu}{2}$ fields and to write down transformation rules for Σ, v^μ, a^μ under infinitesimal transformations.

We find

$$*) \quad \hat{\Sigma} \to \hat{\Sigma} + i[\hat{\epsilon}, \hat{\Sigma}] ; \quad \hat{v}^\mu \to \hat{v}^\mu + i[\hat{\epsilon}, \hat{v}^\mu] - \partial^\mu \hat{v}$$

$$\hat{a}_\mu \to \hat{a}_\mu + i[\hat{\epsilon}, \hat{a}_\mu],$$

$$*) \quad \hat{\Sigma} \to \hat{\Sigma} + i\{\hat{c}, \hat{\Sigma}\} ; \quad a^\mu \to a^\mu + i[\hat{c}, \hat{v}^\mu] - \partial^\mu \hat{c}$$

$$\hat{v}_\mu \to \hat{v}_\mu + i[\hat{c}, \hat{a}_\mu],$$

where $\hat{\epsilon}$ & $\hat{c}$ are the vector and axial gauge transform. parameters.

The most general variation of the quark action under these transformations consistent with the vector current conservation was found by W. Bardeen:

$$\delta_{\epsilon} S_q = F(\hat{\epsilon}, \hat{v}, \hat{a})$$

$$F(\hat{\epsilon}, \hat{v}^{\mu}, \hat{a}^{\mu}) \equiv \int d^4x \, f\left(\hat{\epsilon}(x), \hat{v}_{\mu}(x), \hat{a}_{\mu}(x)\right];$$

$$f(\ldots) = -\frac{1}{16\pi^2} \epsilon_{\mu\nu\alpha\beta} \operatorname{tr}\Big[\hat{\epsilon}(x)\Big[3\hat{v}^{\mu\nu}\hat{v}^{\alpha\beta} + \hat{a}^{\mu\nu}\hat{a}^{\alpha\beta} - 8i\left(\hat{a}^{\mu}\hat{a}^{\nu}\hat{v}^{\alpha\beta} + \hat{a}^{\mu}\hat{v}^{\nu\alpha}\hat{a}^{\beta} + \hat{v}^{\mu\nu}\hat{a}^{\alpha}\hat{a}^{\beta}\right) - 32\hat{a}^{\mu}\hat{a}^{\nu}\hat{a}^{\alpha}\hat{a}^{\beta}\Big]\Big].$$

The consistency with the vector current conservation follows from the fact that the above expression is independent of $\hat{\epsilon}$. (HW)

Now, we need to find the action written in terms of Goldstone fields Σ, such that

$$\delta_{\epsilon} S_{WZW}(\Sigma, \hat{v}, \hat{a}) \equiv F(\hat{\epsilon}, \hat{v}, \hat{a}).$$

It is not easy to find S_{WZW}, so I'll just quote the result.

Problem 2

$$\boxed{S_{WZW} = \int_0^1 ds \, F\left(\frac{\hat{\pi}}{F}, \hat{v}^{\mu}(s), \hat{a}^{\mu}(s)\right),}$$

where $\hat{v}^{\mu}(s)$ and $\hat{a}^{\mu}(s)$ are defined indirectly through

$$\hat{v}^{\mu}(s) = \frac{\ell^{\mu}(s) + r^{\mu}(s)}{2};$$

$$\hat{a}^{\mu}(s) = \frac{\ell^{\mu}(s) - r^{\mu}(s)}{2};$$

$$\ell^{\mu}(s) = \xi^{+}(1-s)\,\ell^{\mu}\,\xi(1-s) - i\,\xi^{+}(1-s)\,\partial^{\mu}\big(\xi(1-s)\big)$$

$$r^{\mu}(s) = \xi^{+}(1-s)\,r^{\mu}\,\xi(1-s) - i\,\xi(1-s)\,\partial^{\mu}\big(\xi^{+}(1-s)\big),$$

where $\xi(s) = e^{i\hat{\pi}/F \cdot s}$.

The WZW action provides "anomaly contributions" to various low-energy processes that ~~contain~~ involve Goldstone bosons and (e.g.) photons.

Let us use this action to compute the decay amplitude for π^0 to 2 photons. To this end, we take $a^{\mu} = 0$ and $\hat{v}^{\mu} = e\hat{Q}A^{\mu}$, where $\hat{Q}$ is the charge matrix for the $SU(3)$ flavor.

We also need to work to lowest order in Goldstone fields. So that

$$\xi(s) \simeq 1 + \frac{i\hat{\pi}}{F}s + O(\pi^2) \to 1, \text{ since}$$

$$F\left(\frac{\hat{\pi}}{F}, \dots\right) \sim O\left(\frac{\pi}{F}\right) \Rightarrow$$

$$F\left(\frac{\pi}{F}, v^{\mu}(s), a^{\mu}(s)\right) \simeq F\left(\frac{\hat{\pi}}{F}, v^{\mu}(0), \emptyset\right) =$$

HW2.

$$= -\frac{1}{16\pi^2}\,\epsilon_{\mu\nu\alpha\beta}\int d^4x\, \mathrm{tr}\left(\frac{\hat{\pi}}{F}\, 3\,\hat{v}^{\mu\nu}\,\hat{v}^{\alpha\beta}\right).$$

Calculating the trace, one finds the interaction Lagrangian for $\pi^0 \to \gamma\gamma$ and $\eta^0 \to \gamma\gamma$.