#7 Chiral Lagrangian for the SU(3)

In the real world, there are three light quarks with masses smaller than the QCD scale ~1 GeV. The other three quarks are heavier and we consider them to be irrelevant at low energies. With this in mind, the QCD Lagrangian becomes

$$\mathcal{L}_{QCD}^{(3)} = \sum_{i\in\{u,d,s\}} \left(\bar{\psi}_L^i i\hat{D}\psi_L^i + \bar{\psi}_R^i i\hat{D}\psi_R^i + (m_i \bar{\psi}_L^i \psi_R^i + h.c.)\right)$$

Neglecting the quark masses, we find the $SU(3)_L \otimes SU(3)_R$ flavor symmetry. At low energies we expect this symmetry to be broken to $SU(3)_{L+R}$, i.e. a diagonal subgroup of $SU(3)_L \otimes SU(3)_R$. This breaking should produce Goldstone bosons and we would like to describe them by generalizing the $SU(2)$ construction that we described in the previous lectures.

We then write $\Sigma = e^{i\frac{\pi^a \lambda^a}{F}}$,

where λ^a are Gell-Mann matrices

$$\lambda_1 = \begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix} \quad \lambda_2 = \begin{bmatrix} 0&-i&0\\i&0&0\\0&0&0 \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{bmatrix}$$

$$\lambda_4 = \begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix} \quad \lambda_5 = \begin{bmatrix} 0&0&-i\\0&0&0\\i&0&0 \end{bmatrix} \quad \lambda_6 = \begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix}$$

$$\lambda_7 = \begin{bmatrix} 0&0&0\\0&0&-i\\0&i&0 \end{bmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{bmatrix}.$$

The Gell-Mann matrices are related to generators of the SU(3) ~~the~~ group T^a:
$T^a = \frac{\lambda^a}{2}$, and $[\lambda^a, \lambda^b] = 2i f^{abc} \lambda^c$, and

$$Tr[\lambda^a \lambda^b] = 2\delta^{ab}.$$

If we write $\sum_{a=1}^{8} \pi^a \lambda^a$ explicitly, we find

$$\sum_{a=1}^{8} \pi^a \lambda^a = \begin{bmatrix} \pi_3 + \pi_8/\sqrt{3} & \pi_1 - i\pi_2 & \pi_4 - i\pi_5 \\ \pi_1 + i\pi_2 & -\pi_3 + \frac{\pi_8}{\sqrt{3}} & \pi_6 - i\pi_7 \\ \pi_4 + i\pi_5 & \pi_6 + i\pi_7 & \frac{-2\pi_8}{\sqrt{3}} \end{bmatrix} =$$

$$= \begin{bmatrix} \pi_0 + \eta/\sqrt{3} & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi_0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\bar{K}^0 & -2\eta/\sqrt{3} \end{bmatrix}.$$

Let's ~~[illegible]~~ find the general transformation rules for the eight Goldstone bosons.
~~We~~ ~~[illegible]~~ Similar to the SU(2) case discussed earlier, we require $\Sigma \to L \Sigma R^+$ under a generic $SU(3)_L \otimes SU(3)_R$ transformation.

What happens under an~~d~~ $SU(3)_{L+R}$ transformation?
In that case, $R = L$, so that

$$\Sigma \to \Sigma' = \exp\left[i\frac{\pi'^a \lambda^a}{F}\right] = L \exp\left[i\frac{\pi^a \lambda^a}{F}\right] L^+$$

Expand both sides of that equation in series of $\frac{\pi'^a \lambda^a}{F}$ or $\frac{\pi^a \lambda^c}{F}$. Consider n-th term:

$$L\left(\frac{\pi^a \lambda^a}{F}\right)^n L^+ = L \underbrace{\frac{\pi^a \lambda^a}{F} \cdot \frac{\pi^a \lambda^a}{F} \cdot \ldots \frac{\pi^a \lambda^a}{F}}_{n \text{ times}} L^+ =$$

$$= L\frac{\pi^a\lambda^a}{F}L^+ \cdot L\frac{\pi^a\lambda^a}{F}L^+ \cdot L\frac{\pi^a\lambda^a}{F}L^+ \cdots L\frac{\pi^a\lambda^a}{F}L^+$$

$$= \left(L\frac{\pi^a\lambda^a}{F}L^+\right)^n \Rightarrow \boxed{\pi'^a\lambda^a = L\frac{\pi^a\lambda^a}{F}L^+}$$

This transformation rule proves that the eight Goldstone bosons transform linearly under $SU(3)_{L+R}$ and represent <u>an octet</u>.

We can also construct an object that transforms as an octet from the quark fields. Taking $\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, we write

$(\hat{\pi}_q)_{ij} = \psi_i \otimes \bar{\psi}_j - \frac{1}{3}(\bar{\psi}\cdot\psi)\delta_{ij}$, where the last term is needed to ensure that $\hat{\pi}_q$ is traceless. Since entries in $\pi^a\lambda^a$ and $\hat{\pi}_q$ have the same "$SU(3)$ quantum numbers", we can read-off the quark content of Goldstone mesons from such a comparison.

We find (examples, not complete list)

$(\hat{\pi}_q)_{12} = u\bar{d} \sim \pi^+$ $\quad (\hat{\pi}_q)_{11} - (\hat{\pi}_q)_{22} \sim \pi_0 \sim$

$(\hat{\pi}_q)_{21} = d\bar{u} \sim \pi^-$ $\quad \sim u\bar{u} - d\bar{d}$

$\eta \sim u\bar{u} + d\bar{d} - 2s\bar{s}$.

On the other hand, under pure axial transformations, the transformation rules for Goldstone bosons are different.

We take $L = e^{i\theta^a\lambda^a}$ and $R = L^+ = e^{-i\theta^a\lambda^a}$

Then $\Sigma \to \Sigma' = L e^{\frac{i\pi^a\lambda^a}{F}} R^+ = L e^{\frac{i\pi^a\lambda^a}{F}} L$.

We can check ~~&~~ what happens for the infinitesimal transformations and small fields. We find:

$$1+i\frac{\pi'^a\cdot\lambda^a}{F} \simeq (1+i\theta_b\lambda^b)\left(1+\frac{i\pi^a\lambda^a}{F}\right)(1+i\theta_c\lambda^c) \Rightarrow$$

$$\boxed{\pi'^a \simeq \pi^a + 2F\theta_a + O(\theta,\pi)}$$

The non-linear nature of the transformation is evident and is essential for π^a's being the Goldstone bosons. ("shift invariance").

Having the matrix Σ, we can easily construct the "kinetic" term of the chiral Lagrangian. Indeed, it is the exact copy of the construction that we had in case of SU(2). We write

$$\mathcal{L}_{O(E^2)} = \frac{F^2}{4}\,\mathrm{Tr}\left(\partial_\mu\Sigma\,\partial^\mu\Sigma^+\right).$$

This Lagrangian describes physics of massless Goldstone bosons and, as we know well, actual π's and K's and η's are not massless. We will now try to ~~construct~~ construct the mass term for the SU(3) chiral Lagrangian assuming that the major source of ~~the~~ masses for the Goldstone bosons ~~is~~ the explicit $L\otimes R$ symmetry breaking mass term in the Lagrangian $\mathcal{L}^{(3)}_{QCD}$.

The mass term reads:

$$\mathcal{L}^{(3)}_{QCD, mass} = \sum_{i \in (u,d,s)} \left(m_i \bar{\Psi}^i_L \Psi^i_R + h.c. \right) = \bar{\Psi}_L \hat{M} \Psi_R + \bar{\Psi}_R \hat{M}^{+} \Psi_L,$$

where $\hat{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$ is the mass matrix.

The $\mathcal{L}^{(3)}_{QCD, mass}$ is not invariant under $SU(3)_L \otimes SU(3)_R$ but it could have been invariant provided that the matrix $\hat{M}$ transforms as $M \to LMR^{+}$.

Suppose now that the mass term in $\mathcal{L}_{O(E^2)}$ is linear in $\hat{M}$ (think about expanding in M and truncating at the first ~~one~~ term). Then, the goal is to write a mass term that is linear in $\hat{M}$ and is invariant under $\Sigma \to L \Sigma R^{+}$ and $M \to LMR^{+}$.

There is basically one term that we can write (we also need to minimize number of derivatives)

$$\mathcal{L}^{mass}_{O(E^2)} = B_0 \, Tr\left(\Sigma^{+} \hat{M}\right) + h.c.$$

The full Lagrangian becomes:

$$\boxed{\mathcal{L}_{O(E^2)} = \frac{F^2}{4} Tr\left(\partial_\mu \Sigma \, \partial^\mu \Sigma^{+}\right) + B_0 \, Tr\left(\Sigma^{+} \hat{M}\right)}$$

We will now try to see what this Lagrangian means.

We do this in an usual way by expanding around small fields and keeping quadratic terms only. We then find:

$$*)\ \frac{F^2}{4}\mathrm{Tr}\left(\partial_\mu \Sigma\, \partial^\mu \Sigma^+\right) \to \frac{F^2}{4}\mathrm{Tr}\left(\frac{\partial_\mu \hat{\pi}\, \partial^\mu \hat{\pi}}{F^2}\right) = \sum_{a=1}^{8} \frac{(\partial_\mu \pi^a)^2}{2}.$$

$$**)\ B_0 \mathrm{Tr}\left(\Sigma^+ \hat{M}\right) + h.c \to B_0 \left[\left(\mathrm{Tr}\left(-i\pi^a \lambda^a \hat{M}\right) + h.c\right)\right.$$

$$\left. - \left[\frac{B_0}{2F^2}\mathrm{Tr}\left(\hat{\pi}^2 M\right) + h.c.\right]\right] = -\frac{B_0}{2F^2}\left(\mathrm{Tr}\left(\hat{\pi}\hat{\pi}\hat{M}\right) + h.c.\right)$$

$$= -\frac{2B_0}{F^2}\mathrm{Tr}\left(\lambda_a \lambda_b \hat{M}\right)\frac{\pi_a \pi_b}{2} = -m^2_{ab}\frac{\pi_a \pi_b}{2}.$$

Hence the mass matrix for the Goldstone bosons is given by

$$\boxed{m^2_{ab} = \frac{2B_0}{F^2}\mathrm{Tr}\left(\lambda_a \lambda_b \hat{M}\right)}$$

One can show that m^2_{ab} is ~~symmetric~~ diagonal, except for the possible off-diagonal entries $\{m^2_{38}$ and $m^2_{83}\}$ (HW). Using this, we first compute a few diagonal contributions to m^2_{ab} and express the results in terms of meson masses:

$$m^2_{11} = \frac{2B_0}{F^2}\mathrm{Tr}\left[\lambda_1^2 \hat{M}\right] = \frac{2B_0}{F^2}\mathrm{Tr}\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\hat{M}\right] \Rightarrow$$

$$m^2_{11} = \frac{2B_0}{F^2}(m_u + m_d)$$

$$m^2_{22} = \frac{2B_0}{F^2}\mathrm{Tr}\left[\lambda_2^2 \hat{M}\right] = \frac{2B_0}{F^2}\mathrm{Tr}\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\hat{M}\right] \Rightarrow$$

$$m^2_{22} = \frac{2B_0}{F^2}(m_u + m_d)$$

Since $m_{12}^2 = m_{21}^2 = 0$ and $\pi^{\pm} = \frac{\Pi_1 \mp i\Pi_2}{\sqrt{2}}$,

we find $\boxed{m_{11}^2 = m_{22}^2 = m_{\pi^\pm}^2 = \frac{2B_0}{F^2}(m_u + m_d)}$

Similar calculations give us masses of charged and neutral k-mesons:

$$\boxed{m_{K^\pm}^2 = \frac{2B_0}{F^2}(m_u + m_s) \qquad m_{K_0}^2 = m_{\bar{K}_0}^2 = \frac{2B_0}{F^2}(m_d + m_s)}$$

What remains to be understood is the $\Pi_3 - \Pi_8$ sector. We find, by an explicit computation;

$$m_{33}^2 = \frac{2B_0}{F^2}(m_u + m_d) \qquad m_{88}^2 = \frac{2B_0}{3F^2}(m_u + m_d + 4m_s)$$

$$m_{38}^2 = \frac{2B_0}{\sqrt{3}F^2}(m_u - m_d) \qquad m_{83}^2 = \frac{2B_0}{\sqrt{3}F^2}(m_u - m_d)$$

The presence of offdiagonal entries means that Π_3 & Π_8 are not physical fields; ~~But~~ they need to be rotated to diagonalize the ~~matri~~ mass matrix.

To diagonalize the mixing matrix, assuming the mixing is small, we can write $[\Pi_3 \simeq \Pi_0 + \eta_0 \theta, \ \Pi_8 \simeq \eta_0 - \theta \Pi_0]$ so that the mixing angle is fixed to

$$\theta \simeq \frac{m_{38}^2}{m_{88}^2 - m_{33}^2} \simeq \frac{m_u - m_d}{(m_u + m_d + 4m_s)}.$$

Since $m_u \sim m_d \sim$ few MeV and $m_s \sim 100$ MeV, $\theta \ll 1$. Neglecting it completely, we find:

$$\boxed{m_{\pi_0}^2 = \frac{2B_0}{F^2}(m_u + m_d),} \quad \text{and} \quad \boxed{m_\eta^2 = \frac{2B_0}{3F^2}(m_u + m_d + 4m_s).}$$

Comparing masses of different mesons, it is easy to see that

$$\boxed{3m_\eta^2 + m_{\pi_0}^2 = 2m_{K_+}^2 + 2m_{K_0}^2}$$

This relation is known as <u>Gell-Mann-Okubo</u> relation. Numerically, if we take $m_\eta = 548$ MeV, $m_{\pi_0} = 135$ MeV, $m_{K_+} = 494$ MeV, $m_{K_0} = 498$ MeV, we find

$$\begin{cases} 3m_\eta^2 + m_{\pi_0}^2 = 0.919 \sim \text{GeV}^2 \\ 2(m_{K^+}^2 + m_{K_0}^2) \simeq 0.984 \sim \text{GeV}^2 \end{cases}$$

Therefore, the Gell-Mann-Okubo relation is accurate ~~[illegible]~~ to about 6%!

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Another interesting information that we can get from these formulas are ratios of quark masses, but this will require us to discuss one more topic – electromagnetism.

What we would like to understand is how to account for a potential electromagnetic mass differences between charged and neutral mesons.

To this end, we'll follow what we did for the case of $SU(2)$ and modify the Lagrangian to introduce the electromagnetic interactions:

$$\frac{F^2}{4} \mathrm{Tr}\left(\partial_\mu \Sigma \, \partial^\mu \Sigma^+\right) \to \frac{F^2}{4} \mathrm{Tr}\left((D_\mu \Sigma)(D^\mu \Sigma)^+\right),$$

where $D_\mu \Sigma \equiv \partial_\mu \Sigma + ie A_\mu [\hat{Q}, \hat{\Sigma}]$.

The "electric charge" matrix $\hat{Q}$ reads:

$$\hat{Q} = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}.$$

We can write $\hat{Q} = \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8$.

Then, looking up the structure constants of $SU(3)$, one concludes that $\hat{Q}$ commutes with λ_3, λ_8, λ_6 and λ_7. Since A_3, A_8, A_6 & A_7 correspond to $\pi_0, \eta, K_0, \overline{K_0}$, we see that neutral particles do not interact with electromagnetic fields to first order in the EM coupling.

It is also easy to see that all other particles ($\pi^\pm, K^\pm$) interact with the same strength (this

means $\pi^\pm$, $K^\mp$ charges are the same,
up to a sign). Hence, we write the following anzats for meson masses:

$$m_{\pi^\pm}^2 = \frac{2B_0}{F^2}(m_u + m_d) + \Delta m_{em}^2 \qquad m_{K_\pm}^2 = \frac{2B_0}{F^2}(m_u + m_s) + \Delta m_{em}^2$$

$$m_{\pi_0}^2 = \frac{2B_0}{F^2}(m_u + m_d) \qquad m_{K_0}^2 = \frac{2B_0}{F^2}(m_d + m_s)$$

$$m_\eta^2 = \frac{2B_0}{3F^2}(m_u + m_d + 4m_s)$$

We find:

$$\boxed{\begin{aligned} \frac{m_d}{m_s} &= \frac{m_{K_0}^2 - m_{K_\pm}^2 + m_{\pi_\pm}^2}{m_{K_0}^2 + m_{K_\pm}^2 - m_{\pi_\pm}^2} \approx 5 \times 10^{-2} \\ \frac{m_d}{m_u} &= \frac{-m_{K_\pm}^2 + m_{K_0}^2 + m_{\pi_\pm}^2}{m_{K_\pm}^2 - m_{K_0}^2 - m_{\pi_\pm}^2 + 2m_{\pi_0}^2} \simeq 1{,}805 \end{aligned}}$$

As follows from these formulas, we obtain ratios of masses without any reference to the η-meson! Hence, we can turn this around and predict the η-meson' mass.. Since $m_s \gg m_d, m_u$,

We find $\underline{m_\eta^2} \simeq \frac{2}{3}\frac{B_0}{F^2}4 \cdot m_s \simeq \frac{4m_{K_0}^2}{3} \simeq \underline{(566\ \text{MeV})^2}$,

where we used $m_{K_0} \simeq 497$ MeV.

The measured value is $m_\eta \simeq 549$ MeV, which means that our prediction works out quite well.