# Lecture 6 External currents and the chiral Lagrangian

An important shortcoming of our pion theory - the chiral Lagrangian - is that pions only interact between themselves. This - of course - is insufficient if we want to apply this formalism in a broader context (weak & electromagnetic interactions of pions). We will now discuss how the construction of the chiral Lagrangian can be extended. To this end, we will promote the symmetry that we have to gauge symmetry.

Consider the following generalization of the chiral transformations:

$$U \to L(x)\, U\, R^+(x), \quad \text{where } L \in SU(2)_L \text{ and } R \in SU(2)_R.$$

Clearly $\mathrm{Tr}\left((\partial_\mu U)(\partial^\mu U^+)\right)$ is not invariant anymore under $x$-dependent transformations. As usual, we need to promote "regular" derivatives to covariant derivatives.

$$\partial_\mu U = \partial_\mu L(x)\, U\, R^+(x) + L\, \partial_\mu U R^+ + L\, U\, \partial_\mu R^+.$$

The first and the third term are terms that we need to get rid of.

Suppose we write $\partial_\mu \to D_\mu = \partial_\mu + i\hat{l}_\mu$, where $\hat{l}_\mu$ is a matrix-valued, $x$-dep. left field. We would like to have the following (only for left transformations)

$$D_\mu U \to L\, D_\mu U\,, \text{ if } U \to \hat{L} U\,, \quad L(x) \in SU_2(2).$$

To accomplish that, we need:

$$D_\mu U = (\partial_\mu + i\hat{\ell}_\mu) U \to (\partial_\mu + i\hat{\ell}'_\mu)\hat{L} U =$$
$$= \hat{L}\,\partial_\mu \hat{U} + (\partial_\mu \hat{L}) U + i\hat{\ell}'_\mu \hat{L} U.$$

To ensure that $D_\mu U$ derivative transforms properly, we need $\underline{\hat{\ell}'_\mu = \hat{L}\,\ell_\mu \hat{L}^+ + i(\partial_\mu L) L^+}$.

Extending this to $SU(2)_R$ transformations is easy. We need

$$\underline{\partial_\mu U \to D_\mu U = \partial_\mu U + i\hat{\ell}_\mu U - i\hat{U} r_\mu}$$

and $\underline{r_\mu \to \hat{R}\, r_\mu \hat{R}^+ + i(\partial_\mu R) R^+}$, under $SU(2)_R$ transformations.

The left- and right currents $\ell_\mu$ & $r_\mu$ can be any of the currents we know – for example weak or electromagnetic.

Let's focus on <u>the electromagnetic interactions</u>

Photons couple to left- and right- degrees of freedom with the same strength.

Therefore, for us, $\ell^\mu = r^\mu = e\hat{Q}A^\mu$, where $\hat{Q} = [\,1/6 + \frac{1}{2}\hat{\tau}_3\,]$ (cf. the quark case)

Therefore $$\boxed{D_\mu U = \partial_\mu U + \frac{ie A_\mu}{2}[\tau_3, U]}$$

The Lagrangian becomes

$$\mathcal{L}^{(2)} = \frac{F_\pi^2}{4} \mathrm{Tr}\left[ D_\mu U (D^\mu U)^+ \right],$$ where

as we said, we will only keep electromagnetic interactions.

Let us start by calculating the electromagnetic current. We can do that by calculating the derivative (c.f. QED)

$$J^\mu_{em} = - \frac{\delta \mathcal{L}^{(2)}}{\delta(eA_\mu)}\Bigg|_{A_\mu = 0} .$$

We find

$$J^\mu_{em} = - \frac{i F_\pi^2}{8} \mathrm{Tr}\left[ [\tau_3, U](\partial_\mu U)^+ + (\partial_\mu U)[\tau_3, U^+] \right].$$

To calculate the trace, let us write

$$U = A + i\vec{B}\cdot\vec{\tau}, \text{ so that } [\tau_3, U] = iB^a[\tau_3, \tau^a] =$$
$$= iB^a\, i2\varepsilon^{3ac}\tau^c =$$
$$= -2[\vec{B}\times\vec{\tau}]_3 ;$$

$$\partial_\mu U^+ = A_{,\mu} - i\vec{B}_{,\mu}\cdot\vec{\tau} .$$

The result: $\mathrm{Tr}\left[ -2[\vec{B}\times\vec{\tau}]_3 (A_{,\mu} - i\vec{B}_{,\mu}\cdot\vec{\tau}) \right]$

$$\equiv 2i\, \mathrm{Tr}\left[ [\vec{B}\times\vec{\tau}]_3 (\vec{B}_{,\mu}\cdot\vec{\tau}) \right] =$$

$$= 4i\, [\vec{B}\times\vec{B}_{,\mu}]_3 .$$

Therefore

$$J^\mu_{em} = \frac{-iF_\pi^2}{8} 4i\, [\vec{B}\times\vec{B}_{,\mu}]_3 \times 2 = F_\pi^2 [\vec{B}\times\vec{B}_{,\mu}]_3 .$$

Now, $\vec{B} = \frac{\vec{\pi}}{|\vec{\pi}|} \sin\left(\frac{|\vec{\pi}|}{F_\pi}\right)$

$\vec{B}_{,\mu} = \partial_\mu \vec{B}$. We need to differentiate $\vec{\pi}$, otherwise there will be no contribution to $J^M_{em}$. Hence $\vec{B}_{,\mu} \to \frac{\partial_\mu \vec{\pi}}{|\vec{\pi}|} \sin\left(\frac{|\vec{\pi}|}{F_\pi}\right) \Rightarrow$

$$J^M_{em} = \frac{2F_\pi^2}{2} \frac{1}{|\vec{\pi}|^2} \sin^2\left(\frac{|\vec{\pi}|}{F_\pi}\right) [\vec{\pi} \times \partial_\mu \vec{\pi}]_3$$

Since $\frac{1}{x^2}\sin^2(x) \underset{x\to 0}{\approx} 1 - \frac{x^2}{3} + \ldots$, we find

$$\boxed{J^M_{em} = \cancel{\phantom{2}} [\vec{\pi} \times \partial_\mu \vec{\pi}]_3 \left(1 - \frac{\vec{\pi}^2}{3F_\pi^2}\right)}$$

A similar calculation can be repeated for the $\mathcal{L}^{(4)}$ Lagrangian. We find:

$$J^M_{em} = (\vec{\pi} \times \partial_\mu \vec{\pi})_3 \left(1 - \frac{\vec{\pi}^2}{3F_\pi^2} + \left[16\alpha_4^{(2)} + 8\alpha_5^{(2)}\right]\frac{m_\pi^2}{F_\pi^2}\right) + \frac{4\alpha_9^{(2)}}{F_\pi^2} \partial^\nu (\partial^M \vec{\pi} \times \partial^\nu \vec{\pi})_3 .$$

Suppose that we want to study how pions interact with the electromagnetic currents. This is - again - can be described by the form factors:

Consider $\langle \pi_1(p_1)\, \pi_2(p_2) | J^{\mu}_{em} | 0 \rangle$.
Lorentz invariance requires:

$$\langle \pi_1(p_1)\, \pi_2(p_2) | J^{\mu}_{em} | 0 \rangle = i G_1(q^2)(p_1 - p_2)^{\mu} + i G_2(q^2)(p_1+p_2)^{\mu}$$

The current is conserved $(q = (p_1+p_2))$ $q_{\mu} J^{\mu}_{em} = 0$
$\Rightarrow$ $G_2(q^2) = \emptyset$. Hence

$$\boxed{\langle \pi_1(p_1)\, \pi_2(p_2) | J^{\mu}_{em} | 0 \rangle \equiv i G_1(q^2)(p_1 - p_2)^{\mu}.}$$

$G_1(q^2)$ is the pion electromagnetic form factor.

We can now calculate $G_1(q^2)$ starting from the expression of $J^{\mu}_{em}$. We find, at leading order in $1/F_{\pi}$:

$$\langle \pi_1(p_1)\, \pi_2(p_2) | \vec{\pi} \times (\partial_{\mu} \vec{\pi})_3 | 0 \rangle =$$
$$= \langle \pi_1(p_1)\, \pi_2(p_2) | \varepsilon^{123} \pi_1 \partial_{\mu} \pi_2 - \varepsilon^{123} \pi_2 \partial_{\mu} \pi_1 | 0 \rangle$$

I'll fix normalization once and for all using
$$\langle \pi^a | \partial_{\mu} \pi^b | 0 \rangle = -i p_{\mu} \delta^{ab} \qquad \langle \pi^a | \pi^b | 0 \rangle = \delta^{ab},$$
so that we obtain

$$\langle \pi_1(p_1)\, \pi_2(p_2) | \vec{\pi} \times (\partial_{\mu} \pi)_3 | 0 \rangle = i (p_1 - p_2)_{\mu}.$$

This means that at leading order in $1/F_{\pi}$, the form-factor $G_1(q^2) \equiv 1$.

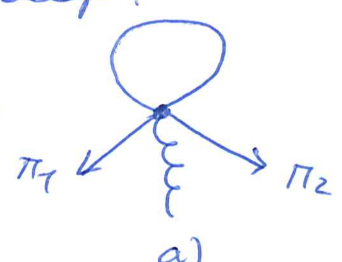
Now, suppose we want to continue with this calculation and account for subleading terms in $J^{\mu}_{em}$. Then, it is easy to see that there are 2 types of terms: terms that are quadratic in the fields and terms that are quartic.

The quadratic terms is easy to deal with. Calculation similar to what we first did gives

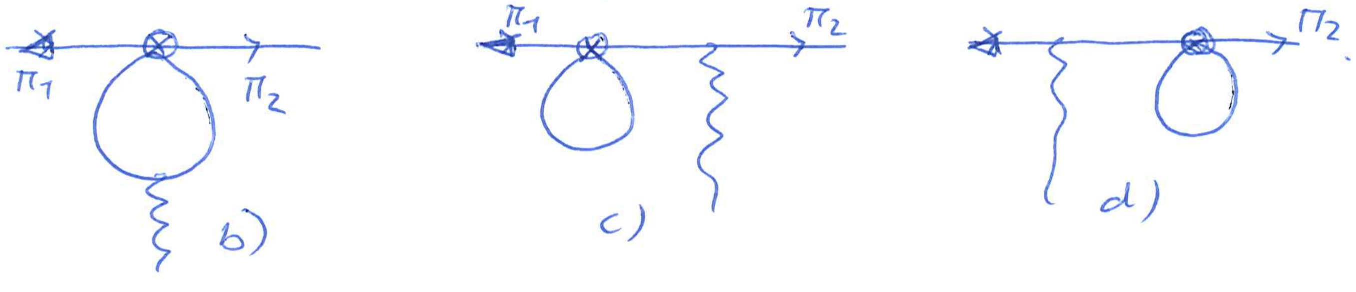
$$\delta G_1(q^2) = \left[16\,\alpha_4^{(2)} + 8\alpha_5^{(2)}\right]\frac{m_\pi^2}{F_\pi^2} + \frac{2\alpha_9^{(2)}}{F_\pi^2}\,q^2.$$

~~Re~~ Terms with 4 pions require that we get read of two pion fields; we can do this forming a loop!

$$\Delta J_\mu^{em} = \left(\vec{\pi}\times\partial_\mu\vec{\pi}\right)_3\left(-\frac{\vec{\pi}^2}{3F_\pi^2}\right) \longleftrightarrow$$




Of course, we can also use $A_\mu$-independent part of $\mathcal{L}^{(2)}$ to develop perturbation theory for the form factor. Since $\mathcal{L}^{(2)}$ provides a 4-pion scattering vertex, we ~~too~~ have




Of course, diagrams c) & d) are not relevant at this point since they are 1-particle irreducible; so we need to understand a) & b)

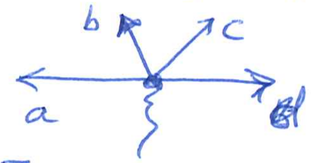
To compute Feynman diagram a) we need the Feynman rule for the vertex.
We calculate

$$\langle \pi^a \pi^b \pi^c \pi^d | \Delta J^{em}_\mu | 0 \rangle = -\frac{1}{3F_\pi^2} \langle \pi^a \pi^b \pi^c \pi^d | \varepsilon^{3mn} \pi^m \partial_\mu \pi^n \delta_{kl} \pi^k \pi^l | 0 \rangle =$$

$$= -\frac{1}{3F_\pi^2} \varepsilon^{3mn} \delta^{kl} \langle \pi^a \pi^b \pi^c \pi^d | \pi^m \partial_\mu \pi^n \pi^k \pi^l | 0 \rangle.$$

It is now easy to see that this vertex can be written as:

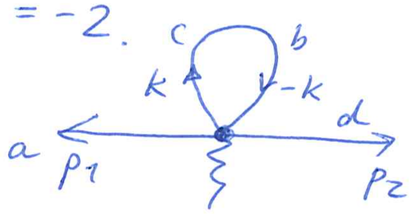
$$\langle \pi^a \pi^b \pi^c \pi^d | \Delta J^{em}_\mu | 0 \rangle = \frac{Xi}{3F_\pi^2} \times \Big\{ \epsilon_{3ab} \delta_{cd} (p_a - p_b)^\mu + \epsilon_{3ac} \delta_{bd} (p_a - p_c)^\mu + \varepsilon_{3ad} \delta_{bc} (p_a - p_d)^\mu + \epsilon_{3bc} \delta_{ad} (p_b - p_c)^\mu + \varepsilon_{3bd} \delta_{ac} (p_b - p_d)^\mu + \varepsilon_{3cd} \delta_{ab} (p_c - p_d)^\mu \Big\} \equiv$$




$$\equiv \equiv V^\mu_{abcd} [p_a, p_b, p_c, p_d]$$

To calculate $X$, we take $a=1, b=2, c=3, d=3$.
Then $\langle \pi^1 \pi^2 \pi^3 \pi^3 | \Delta J^{em}_\mu | 0 \rangle = 2i \epsilon_{312} (p_2 - p_1)_\mu$.)
so that $X = -2$.

To calculate




, we write

$$Dia_a \equiv \mu^{d-4} \int \frac{d^d k}{(2\pi)^4} \frac{i \delta^{bc}}{k^2 - m_\pi^2} V^\mu_{1bc2} (p_1, k, -k, p_2).$$

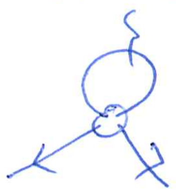
A straightforward algebra gives

$$\boxed{Dia_a = \frac{10}{3F_\pi^2}(p_1-p_2)^\mu I_0(m_\pi^2)}\text{, where}$$

$$I_0(m_\pi^2) = \int \frac{d^dk}{(2\pi)^d}\frac{\mu^{d-4}}{k^2-m_\pi^2} = \frac{-i m_\pi^2}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\left(\frac{\mu}{m}\right)^{2\varepsilon}.$$

Note that, because of the dimensional regularization, the loop contributes $O\left(\frac{m_\pi^2}{F_\pi^2}\right)$ correction to the form-factor in spite of the fact that loop integral quadratically diverges in UV.

— The second contribution is from the diagram b)



We obtain, after some algebra:

$$\boxed{Dia_b = -\frac{4\mu^{2\varepsilon}}{F_\pi^2}\frac{(p_1-p_2)^\mu}{d}\int_0^1 dx_1 dx_2\,\delta(1-x_1-x_2) \times \frac{-i}{(4\pi)^{d/2}}\frac{\Gamma(1-d/2)\Gamma(d/2+1)}{\Gamma(d/2)}\left(m_\pi^2-\frac{q^2}{4}\left(1-(x_1-x_2)^2\right)\right)^{1-\varepsilon}}.$$

It is easy to see that both contributions: $Dia_a$ & $Dia_b$ are divergent. The divergencies can be extracted in a straightforward way: (repeat $z\Gamma(z)=\Gamma(1+z)$)

$$Dia_a = \frac{10}{3F_\pi^2}(p_1-p_2)^\mu \frac{i m_\pi^2}{(4\pi)^{d/2}}\left\{\frac{1}{\varepsilon}+\ln\frac{\mu^2}{m_\pi^2}+\dots\right\}$$

$$Dia_b = \frac{2i(p_1-p_2)^\mu}{F_\pi^2(4\pi)^{d/2}}\left\{-1/\varepsilon+\dots\right\}\left(m_\pi^2-\frac{q^2}{6}\right)$$

Combining these results with the contribution of $\mathcal{L}^{(4)}$ to $\delta G_1$, we obtain:

$$\delta G_1(q^2) = \left[16\alpha_4^{(2)} + 8\alpha_5^{(2)}\right]\frac{m_\pi^2}{F_\pi^2} + \frac{2\alpha_9^{(2)}}{F_\pi^2}q^2$$

$$+ \left[\frac{10}{3(4\pi)^{d/2}}\times\frac{1}{\varepsilon} - \frac{2}{(4\pi)^{d/2}}\frac{1}{\varepsilon}\right]\frac{m_\pi^2}{F_\pi^2} + \frac{1}{3(4\pi)^{d/2}}\frac{1}{\varepsilon}\frac{q^2}{F_\pi^2} + \ldots \Rightarrow$$

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$$\delta G_1 \equiv \left[16\alpha_4^{(2)} + 8\alpha_5^{(2)} + \frac{4}{3(4\pi)^{d/2}}\frac{1}{\varepsilon}\right]\frac{m_\pi^2}{F_\pi^2} + \left(2\alpha_9^{(2)} + \frac{1}{3(4\pi)^{d/2}}\frac{1}{\varepsilon}\right)\frac{q^2}{F_\pi^2}$$

$+$ finite.

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We can remove those divergences by changing parameters in $\mathcal{L}^{(4)}$ from "bare" to renormalized. For example: if $\alpha_9^r = \alpha_9^{(2)} + \frac{1}{6\varepsilon(4\pi)^{d/2}}$, then ~~Eq~~ the $q^2$-dependence of $G^{(1)}$ ~~is~~ becomes finite. Similar to ordinary renormalizable theories, the effective field theories are made finite by adjusting parameters of the Lagrangian. However, in EFT this can and should be done order-by-order in ~~the~~ chiral expansion. Although number of constants that we can adjust increases from order to order in $E/F_\pi$ expansion, ~~as~~ the number of constants is still small compared to the number of measurements that we can desribe.



Finally, we note that $G(q^2=0)=1$ (electromagnetic charge conservation). Therefore $\delta G_1(q^2=0)=0$. Our formulas do not comply with that. The reason is that we did not account for the self-energy corrections on external pion legs (i.e. the wave function renormalization). Putting everything together, we find:

$$G_\pi(q^2) = 1 + \frac{2\alpha g^{(2),r}(\mu)}{F_\pi^2} q^2 + \frac{1}{6(4\pi)^2 F_\pi^2}\left[(q^2-4m_\pi^2)\, H\!\left(\frac{q^2}{m_\pi^2}\right) - q^2 \ln\frac{m_\pi^2}{\mu^2} - q^2/3\right],$$

where

$$H\left(\frac{q^2}{m_\pi^2}\right) = -\int_0^1 dx \ln\left(1-\frac{q^2}{m_\pi^2}x(1-x)\right)$$

For $q^2 \ll m_\pi^2$, we find

$$G_\pi(q^2) \simeq 1 + \left[\frac{2\alpha g^{(2),r}(\mu)}{F_\pi^2} - \frac{1}{6(4\pi)^2 F_\pi^2}\left(\ln\frac{m_\pi^2}{\mu^2}+1\right)\right] q^2 + \dots$$

The pion form factor is generally parametrized as $G_\pi(q^2) \simeq 1 + \frac{\langle r_\pi^2\rangle}{6} q^2$, where $\langle r_\pi^2\rangle$ is the pion charge radius.

The prediction of the chiral perturbation theory is

$$\boxed{\langle r_\pi^2\rangle = \frac{12\alpha g^{(2),r}(\mu)}{F_\pi^2} - \frac{1}{16\pi^2 F_\pi^2}\left(\ln\frac{m_\pi^2}{\mu^2}+1\right).}$$