Lecture 5. Chiral Lagrangian for $SU(2)$. Pion scattering

We ended up the previous lecture discussing a particular, non-linear realization of the pion fields in the σ-model $\Sigma = \rho\, U$, $U^+ U = 1$, $U = e^{i\vec{\pi}\cdot\vec{\tau}/F_\pi}$ that allows us to remove all the excitations of the field $\rho = (v + \sigma)$ at low energies in a clean way. The resulting Lagrangian was still invariant under $SU(2)_L \otimes SU(2)_R$ transformations of the U-field.

We can now forget about the σ-model alltogether and ask the following question:

what is the most general form of the Lagrangian constructed from U-fields if two conditions are imposed:

1) the U field transforms as $U \to L U R^+$ under $SU(2)_L \otimes SU(2)_R$
2) the Lagrangian is invariant under $SU(2)_L \otimes SU(2)_R$.

Note that the renormalizability of the Lagrangian is not a condition since we try to construct an effective field theory valid at low energies. <u>only</u>.

<u>The first thing</u> we need to realize then is that derivatives of fields (i.e. $\partial_\mu U$) become four momenta p_μ in matrix elements for pion scattering. Since we are interested in low-energy physics, terms with more derivatives (more momenta) are suppressed, relative to terms without (or with small number of) derivatives.

Hence, when constructing the Lagrangian we move from terms with the smallest number of derivatives possible to terms with increasing number of derivatives.

The smallest number of derivatives possible is zero. However, there are no nontrivial terms that we can build from U's that satisfy the two conditions. Indeed, since $UU^+ = \hat{1}$, terms like $Tr(UU^+)$ give irrelevant contributions.

One-derivatives terms are not possible because of Lorentz-invariance.

Two-derivative terms are possible and should contain $Tr((\partial_\mu U)(\partial^\mu U)^+)$ operator. Clearly, another possible operator $Tr((\partial^2 U) U^+)$ is just the

same as the previous one thanks to the integration-by-parts. It is not difficult to convince oneself that $\mathrm{Tr}((\partial_\mu U)(\partial^\mu U)^+)$ is the <u>only</u> 2-derivative operator we need to consider (<u>note that this does require some thinking</u>; for example we can write $\mathrm{Tr}[(\partial_\mu U)\, U^+ (\partial^\mu U)\, U^+]$ and it is not immediately obvious that it is the same as $\mathrm{Tr}[(\partial_\mu U)(\partial^\mu U)^+]$.
But it is: indeed: $U U^+ = 1 \Rightarrow (\partial_\mu U) U^+ = -U \partial_\mu U^+$

$$\Rightarrow \mathrm{Tr}[(\partial_\mu U)\, U^+ (\partial^\mu U)\, U^+] = -\mathrm{Tr}[+U(\partial_\mu U^+)(\partial^\mu U)\, U^+]$$
$$= -\mathrm{Tr}(U^+ U (\partial_\mu U^+)(\partial^\mu U)] = -\mathrm{Tr}[(\partial_\mu U)(\partial^\mu U^+)].)$$

We therefore write:

$$\mathcal{L}_{(2)} \equiv c_2 \cdot \mathrm{Tr}\left((\partial_\mu U)(\partial^\mu U)^+\right),$$ where c_2 is an unknown coefficient.

To fix c_2, note that for small fields $\pi^a/F_\pi \ll 1$, we should recover kinetic term for the pion fields from $\mathcal{L}_{(2)}$.
To this end, write $U = e^{i\vec{\pi}\cdot\vec{\tau}/F_\pi} \approx$

$$\approx 1 + \frac{i\,\vec{\pi}\cdot\vec{\tau}}{F_\pi} + O\left(\left(\frac{\pi}{F_\pi}\right)^2\right)$$

$$\partial_\mu U = i\,\frac{(\partial_\mu \vec{\pi})\cdot\vec{\tau}}{F_\pi} \qquad (\partial_\mu U)^+ = \frac{-i(\partial_\mu\vec{\pi})\cdot\vec{\tau}}{F_\pi}$$

$$\mathrm{Tr}\left((\partial_\mu U)(\partial^\mu U)^+\right) = \frac{1}{F_\pi^2}\,\mathrm{Tr}\left([\partial_\mu\vec{\pi}\cdot\vec{\tau}]\,[\partial^\mu\vec{\pi}\cdot\vec{\tau}]\right)$$

$$= \frac{1}{F_\pi^2} \partial_\mu \pi^a \partial^\mu \pi^b \cdot \mathrm{Tr}[\tau^a \cdot \tau^b] =$$

$$= \frac{1}{F_\pi^2} (\partial_\mu \pi^a)(\partial^\mu \pi^b)\, 2\delta^{ab} = \frac{2(\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi})}{F_\pi^2}$$

(Note that $\vec{\tau}$ here are Pauli matrices).

Therefore, for small fields we find

$$\mathcal{L}_{(2)} \approx \frac{c_2 \cdot 2}{F_\pi^2} (\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi}) \longleftrightarrow \frac{(\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi})}{2}.$$

This implies that $\boxed{c_2 = F_\pi^2/4.}$

Hence, the chiral Lagrangian restricted to terms with just two derivatives is fixed completely:

$$\boxed{\mathcal{L}_{(2)} = \frac{F_\pi^2}{4} \mathrm{Tr}((\partial_\mu U)(\partial^\mu U)^+).}$$

Before we continue with the discussion of the chiral Lagrangian with more derivatives I would like to show you what kind of physics is possible to do with $\mathcal{L}_{(2)}$.

Unfortunately, just working with $\mathcal{L}_{(2)}$ is not very useful for pions since pions are <u>**not** massless</u>. We somehow need a term in the Lagrangian that gives pions the mass. This is tricky since the masslessness of the pion is a consequence of it being a

Goldstone boson; this also leads to the requirement of the Lagrangian being invariant under $SU(2)_L \otimes SU(2)_R$.
We will assume that the symmetry is broken from $SU(2)_L \otimes SU(2)_R \to SU(2)_{L+R}$ by explicit (equal) mass terms for up and down quarks in the fundamental Lagrangian. Then, we can add to $\mathcal{L}_{(2)}$ terms that contain minimal number of derivatives and are invariant under $U \to L U R^+$ only for $L = R$. There is one term like that

$\Delta \mathcal{L}_{(2)} = c_m \cdot Tr[U + U^+]$, where c_m is an unknown coefficient. Again, we find it by considering the weak field limit $(\pi / F_\pi \ll 1)$:

$$U \simeq 1 + \frac{i\pi^a \tau^a}{F_\pi} - \frac{1}{2F_\pi^2} \hat{\pi}\cdot\hat{\pi}$$

$$U^+ \simeq 1 - \frac{i\hat{\pi}}{F_\pi} - \frac{1}{2F_\pi^2} \hat{\pi}\cdot\hat{\pi}$$

$$\Rightarrow \quad U + U^+ = 2 - \frac{1}{F_\pi^2} \hat{\pi}\cdot\hat{\pi} \quad \Rightarrow$$

$$Tr(U + U^+) \equiv 4 - \frac{1}{F_\pi^2} Tr(\hat{\pi}\cdot\hat{\pi}) =$$

$$= 4 - \frac{1}{F_\pi^2} \pi^a \pi^b \, 2\delta^{ab} = 4 - \frac{2}{F_\pi^2} \pi^2.$$

Since the mass term should be just $\left(-\frac{m_\pi^2}{2}\right)\vec{\pi}^2$, we find $c_m = \frac{m_\pi^2 F_\pi^2}{4} \Rightarrow$

$$\Delta\mathcal{L}_{(2)} = \frac{m_\pi^2 F_\pi^2}{4} \mathrm{Tr}\left(\mathcal{U}+\mathcal{U}^\dagger\right);$$ combining $\mathcal{L}_{(2)}$ & $\Delta\mathcal{L}_{(2)}$, we obtain:

$$\boxed{\mathcal{L}_{(2)}^{mod} = \frac{F_\pi^2}{4} \mathrm{Tr}\left((\partial_\mu \mathcal{U})(\partial^\mu \mathcal{U})^\dagger\right) + \frac{m_\pi^2 F_\pi^2}{4} \mathrm{Tr}\left(\mathcal{U}+\mathcal{U}^\dagger\right)}$$

Next step is to see what can be done with this Lagrangian? One thing that it contains are - of course - interaction terms for pions and those interaction terms are uniquely determined by the chiral (or L+R) symmetry in a sense that the interactions are described in terms of F_π and m_π <u>only</u>.

Let's compute the interaction terms.

We will do that in a way that will allow us to ~~extend~~ use ~~these~~ intermediate results elsewhere. So let's start with

$$\boxed{\begin{aligned} \mathcal{U} = e^{i\pi^a\tau^a/F_\pi} &\equiv \cos\left(\frac{|\vec{\pi}|}{F_\pi}\right) + i\frac{\vec{\pi}\cdot\vec{\tau}}{|\vec{\pi}|}\sin\left(\frac{|\vec{\pi}|}{F_\pi}\right) \\ &\equiv A + i\vec{B}\cdot\vec{\tau}. \end{aligned}}$$

$$\partial_\mu \mathcal{U} = (\partial_\mu A) + i(\partial_\mu \vec{B})\cdot\vec{\tau} = A_{,\mu} + i\vec{B}_{,\mu}\cdot\vec{\tau},$$

so that $\partial_\mu A = A_{,\mu}$ & $\partial_\mu \vec{B} = \vec{B}_{,\mu}$ notation is introduced.

In terms of that notation:

$$Tr(\mathcal{U}+\mathcal{U}^\dagger) = Tr(2A) \equiv 4A = 4\cos(|\vec{\pi}|/F_\pi)$$

$$Tr(\partial_\mu \mathcal{U}\, \partial_\mu \mathcal{U}^\dagger) = Tr\left((A_{,\mu}+i\vec{B}_{,\mu}\cdot\vec{\tau})(A^{,\mu}-i\vec{B}^{,\mu}\cdot\vec{\tau})\right)$$

$$= 2\left(A_{,\mu}A^{,\mu}+\vec{B}_{,\mu}\cdot\vec{B}^{,\mu}\right).$$

Now, let's calculate $A_{,\mu}$ & $\vec{B}_{,\mu}$:

$$*)\quad A_{,\mu} = -\sin\left(\frac{|\vec{\pi}|}{F_\pi}\right)\frac{(\partial_\mu\pi^a)\cdot\pi^a}{|\vec{\pi}|\,F_\pi} = -\frac{(\partial_\mu\vec{\pi})\cdot\vec{\pi}}{F_\pi\,|\vec{\pi}|}\cdot\sin\left(\frac{|\vec{\pi}|}{F_\pi}\right)$$

$$\vec{B}_{,\mu} = \partial_\mu\left(\frac{\vec{\pi}}{|\vec{\pi}|}\sin\left(\frac{|\vec{\pi}|}{F_\pi}\right)\right) = \frac{\partial_\mu\vec{\pi}}{|\vec{\pi}|}\sin\left(\frac{|\vec{\pi}|}{F_\pi}\right)$$

$$+(\vec{\pi})\left[-\frac{(\partial_\mu\vec{\pi})\cdot\vec{\pi}}{|\vec{\pi}|^3}\sin\left(\frac{|\vec{\pi}|}{F_\pi}\right)+\frac{(\partial_\mu\vec{\pi}\cdot\vec{\pi})}{F_\pi|\vec{\pi}|^2}\cos\left(\frac{|\vec{\pi}|}{F_\pi}\right)\right] \Rightarrow$$

$$*)\quad \vec{B}_{,\mu} = \frac{\partial_\mu\vec{\pi}}{F_\pi}\cdot\frac{F_\pi}{|\vec{\pi}|}\sin\left(\frac{|\vec{\pi}|}{F_\pi}\right)+\frac{\vec{\pi}\cdot[(\partial_\mu\vec{\pi})\cdot\vec{\pi}]}{|\vec{\pi}|^2\cdot F_\pi}\times$$

$$\times\left(\cos\left(\frac{|\vec{\pi}|}{F_\pi}\right)-\frac{F_\pi}{|\vec{\pi}|}\sin\left(\frac{|\vec{\pi}|}{F_\pi}\right)\right).$$

~~Hence~~ We need 1st non-trivial interaction terms, so first terms beyond quadratic.

This implies:

$$*)\quad A_{,\mu} = -\frac{(\partial_\mu\vec{\pi})\cdot\vec{\pi}}{F_\pi^2}\cdot\frac{F_\pi}{|\vec{\pi}|}\sin\left(\frac{|\vec{\pi}|}{F_\pi}\right)\to -\frac{(\partial_\mu\vec{\pi})\cdot\vec{\pi}}{F_\pi^2},$$

since $$\frac{1}{x}\sin(x) \underset{x\to 0}{\simeq} 1-x^2/6+O(x^4).$$

*) For $\vec{B}_{,\mu}$ we'll need an expansion of $\cos(x)-\frac{1}{x}\sin(x)\xrightarrow[x\to 0]{} -\frac{1}{3}x^2$, in addition to the expansion of $1/x\,\sin(x)$.

We find

$$\vec{B}_{,\mu} = \frac{\partial_\mu \vec{\pi}}{F_\pi} \times \left(1 - \frac{\vec{\pi}^2}{6F_\pi^2}\right) - \frac{\vec{\pi}(\partial_\mu \vec{\pi} \cdot \vec{\pi})}{3F_\pi^3}.$$

Hence, we obtain

$$\boxed{\mathcal{L}_{(2)} \to \frac{1}{2}(\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi}) + \frac{1}{6}\frac{(\partial_\mu \vec{\pi} \cdot \vec{\pi})^2}{F_\pi^2} - \frac{1}{6F_\pi^2}(\partial_\mu \vec{\pi})^2 \vec{\pi}^2 + O(\pi^6)}$$

The mass term contribution is obtained upon expanding $\cos\left(\frac{|\vec{\pi}|}{F_\pi}\right)$: $\underline{\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots}$

$\Rightarrow$

$$\Delta\mathcal{L}_{(2)} = \frac{m_\pi^2 F_\pi^2}{4} \mathrm{Tr}(U + U^+) = m_\pi^2 F_\pi^2 \cos\left(\frac{|\vec{\pi}|}{F_\pi}\right) \simeq$$

$$\simeq m_\pi^2 F_\pi^2 \left(1 - \frac{1}{2}\frac{\vec{\pi}^2}{F_\pi^2} + \frac{1}{24}\frac{(\vec{\pi}^2)^2}{F_\pi^4}\right) \Rightarrow$$

$$\boxed{\Delta\mathcal{L}_{(2)} \simeq \underline{m_\pi^2 F_\pi^2} - \frac{m_\pi^2 \vec{\pi}^2}{2} + \frac{m_\pi^2 (\vec{\pi}^2)^2}{24 F_\pi^2} + O(\pi^6)}$$

Putting everything together, we find

$$\boxed{\mathcal{L}_{(2)}^{mod} = \frac{1}{2}(\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi}) - \frac{m_\pi^2}{2}\vec{\pi}^2 + \left[\frac{m_\pi^2 (\vec{\pi}^2)^2}{24 F_\pi^2} + \frac{1}{6}\frac{(\partial_\mu \vec{\pi} \cdot \vec{\pi})^2}{F_\pi^2} - \frac{1}{6F_\pi^2}(\partial_\mu \vec{\pi})^2 \vec{\pi}^2\right] + O(\pi^6)}$$

The term in the square brackets is the interaction term.

We can use the above Lagrangian to calculate pion scattering at low energy.

We consider the process $0 \to \pi^a + \pi^b + \pi^c + \pi^d$, calculate the interaction terms from $\mathcal{L}^{mod}_{(2)}$ and obtain:

$$A_{0\to\pi\pi\pi\pi} = \frac{i}{F_\pi^2}\Big\{ \delta^{ab}\delta^{cd}\left[(p_a+p_b)^2 - m_\pi^2\right] + \delta^{ac}\delta^{bd}\left[(p_a+p_c)^2 - m_\pi^2\right]$$

$$+ \delta^{ad}\delta^{bc}\left[(p_a+p_d)^2 - m_\pi^2\right]\Big\}. \quad (HW).$$

This amplitude is supposed to desribe scattering of pions at low energies; the neglected terms should be suppressed by powers of $\frac{m_\pi}{F_\pi}$ & $\frac{p_a^\mu}{F_\pi}$. Again, we stress that $A_{\pi\pi\to\pi\pi}$ is written entirely in terms of F_π & m_π; no other parameters appear.

The above amplitude follows from the Lagrangian $\mathcal{L}^{mod}_{(2)}$ which contains terms <u>with up to 2 **derivatives**</u>. The effective Lagrangian may, however, contain terms with more than two derivatives, so we would like to discuss those.

In principle, we should write down <u>all possible terms</u> and then remove some of them using integration-by-parts, $SU(2)$ identities and, in principle, equations of motion that follow from $\mathcal{L}^{(2)}$

[we consider $SU(2)_L \otimes SU(2)_R$ situation, $m_\pi \to 0$]

We can write the following terms:

$\mathcal{O}_1 = Tr[((\partial_\mu \partial^\nu U))(\partial^\mu \partial_\nu U)^+]$;

$\mathcal{O}_2 = Tr[\partial_\mu U \partial^\mu U^+] Tr[\partial_\nu U \partial^\nu U^+] = \mathcal{L}_4^{(a)}$

$\mathcal{O}_3 = Tr[\partial_\mu U \partial^\nu U^+] Tr[\partial^\mu U \partial_\nu U^+] = \mathcal{L}_4^{(b)}$

$\mathcal{O}_4 = Tr[\partial_\mu U \partial_\nu U^+] Tr[\partial_\nu U \partial^\mu U^+]$

$\mathcal{O}_5 = Tr[\partial_\mu U \partial^\mu U^+ \partial_\nu U \partial^\nu U^+]$

$\mathcal{O}_6 = Tr[\partial_\mu U \partial^\nu U^+ \partial_\mu U \partial^\nu U^+]$, etc.

In principle, $\mathcal{L}^{(4)}$ can be a linear combination of all these terms.

So the question is - 'to what extent they are independent.

We'll start with $\mathcal{O}^{(1)}$. Integrating by parts we can cast it to a form

$\mathcal{O}_1 \to Tr[(\partial_\nu \partial^\nu U)(\partial_\mu \partial^\mu U)]$

- An operator of this type doesn't need to be considered since it will vanish by virtue of equations of motion of $\mathcal{L}^{(2)}$ [$\partial_\mu \partial^\mu U \equiv \emptyset$].

$\mathcal{O}_4$ is easily reduced to $\mathcal{L}^{(b)}_{(4)}$, by using ~~integration by parts twice~~ $\partial_\mu(UU^\dagger)=0$, we derive $\partial_\nu U\,\partial^\mu U^\dagger \to \partial^\mu U\,\partial_\nu U^\dagger$. This maps $\mathcal{O}_4$ to $\mathcal{L}^{b}_{(4)}$.

$\mathcal{O}_5$ is slightly more tricky; to analyse it, we will use the same notations that we employed for the analysis of pion scattering.

$$U = A + i\vec{B}\cdot\vec{\tau} \qquad \partial_\mu U = A_{,\mu} + i\vec{B}_{,\mu}\cdot\vec{\tau}.$$

$$\partial_\mu U\,\partial^\mu U^\dagger = (A_{,\mu} + i\vec{B}_{,\mu}\cdot\vec{\tau})(A_{,\mu} - i\vec{B}_{,\mu}\cdot\vec{\tau}) =$$

$$= A_{,\mu}\cdot A_{,\mu} + \vec{B}^a_{,\mu}\cdot\vec{B}^b_{,\mu}\,\tau^a\cdot\tau^b$$

Since $\tau^a\cdot\tau^b = \delta^{ab} + i\varepsilon^{abc}\tau^c$ and since $\vec{B}_{,\mu}\times\vec{B}_{,\mu} = 0$, we find

$$\partial_\mu U\,\partial^\mu U^\dagger \equiv A_{,\mu}\cdot A_{,\mu} + \vec{B}_{,\mu}\cdot\vec{B}_{,\mu}.$$

Therefore: $\mathcal{O}_5 = \mathrm{Tr}\left[\partial_\mu U\,\partial^\mu U^\dagger\,\partial_\nu U\,\partial^\nu U^\dagger\right] =$

$$= 2\left[A_{,\mu}\cdot A_{,\mu} + \vec{B}_{,\mu}\cdot\vec{B}_{,\mu}\right]\left[A_{,\nu}\cdot A_{,\nu} + \vec{B}_{,\nu}\cdot\vec{B}_{,\nu}\right]$$

$$= \frac{1}{2}\mathcal{O}_2 = \frac{1}{2}\mathcal{L}^{(a)}_4. \Rightarrow$$

$$\boxed{\mathcal{O}_5 = \frac{1}{2}\mathcal{L}^{(a)}_4}$$

It is slightly more complicated to show that $\mathcal{O}_6$ can be expressed in terms of $\mathcal{L}^{(a)}_4$ & $\mathcal{L}^{(b)}_4$. ~~[illegible]~~ (HW)

The result:

$$*)\quad \mathrm{Tr}\left[\partial_\mu U \partial^\nu U^+ \partial_\mu U \partial^\nu U^+\right] = \frac{1}{2}\mathcal{L}_4^{(b)} - \frac{1}{4}\mathcal{L}_4^{(a)}.$$

It is possible to go through a similar analysis for all other terms with 4 derivatives and to show that they can be expressed through $\mathcal{L}_4^{(a)}$ and $\mathcal{L}_4^{(b)}$.

Hence, the $SU(2)_L \otimes SU(2)_R$ effective Lagrangian to order $\mathcal{O}(E^4)$ is determined by only 3 terms:

$$\boxed{\mathcal{L}_{eff}^{(4)} = \frac{F_\pi^2}{4}\mathrm{Tr}\left[\partial_\mu U \partial^\mu U^+\right] + \underline{\underline{\alpha_1}}\left[\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right)\right]^2 + \underline{\underline{\alpha_2}}\,\mathrm{Tr}\left(\partial_\mu U \partial_\nu U^+\right)\cdot \mathrm{Tr}\left(\partial^\mu U \partial^\nu U^+\right) + \ldots}$$

The two constants - $\underline{\underline{\alpha_1}}$ & $\underline{\underline{\alpha_2}}$ are arbitrary and can not be fixed in the same way as the coefficient of the term with just 2 derivatives.

Note, however, that the situation becomes significantly worse if we include the mass-suppressed terms. We obtain then

$$\mathcal{L} = \frac{F_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right) + \frac{F_\pi^2}{4} m_\pi^2 \mathrm{Tr}\left(U + U^+\right) + \alpha_1\left[\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right)\right]^2 + \alpha_2 \mathrm{Tr}\left[\partial_\mu U \partial^\nu U^+\right]\mathrm{Tr}\left(\partial_\mu U \partial^\nu U^+\right) +$$

$$+ \alpha_4 \, \mathrm{Tr}\left(\partial_\mu \mathcal{U} \partial^\mu \mathcal{U}^+\right) m_\pi^2 \, \mathrm{Tr}\left(\mathcal{U} + \mathcal{U}^+\right)$$

$$+ \alpha_5 \, \mathrm{Tr}\left(\partial_\mu \mathcal{U} \partial^\mu \mathcal{U}^+ (\mathcal{U} + \mathcal{U}^+)\right) m_\pi^2$$

$$+ \alpha_6 \left[\mathrm{Tr}\,(\mathcal{U} + \mathcal{U}^+)\right]^2 m_\pi^4 + \alpha_7 \left[\mathrm{Tr}\,[(\mathcal{U} - \mathcal{U}^+)]\right]^2 m_\pi^4$$

$$+ \alpha_8 \, m_\pi^4 \, \mathrm{Tr}\left(\mathcal{U}^2 + \mathcal{U}^{+2}\right) + O(E^6, m_\pi^6, E^4 m_\pi^2, E^2 m_\pi^2).$$