Lecture 4: Chiral symmetry: linear σ-model

This is an old model of Gell-Mann & Levy, who attempted to describe interactions of protons, neutrons and pions.

Consider a proton & neutron & form an $SU(2)$ doublet $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$. The $SU(2)$ symmetry is an isospin symmetry of strong interactions: if $\psi \to \psi + \delta\psi$, with $\delta\psi = i\varepsilon^a T^a \psi$, $T^a = \frac{\tau^a}{2} \in SU(2)$, the Lagrange density must be invariant.

Now, consider only kinetic term assuming that ψ is a massless field.

$\mathcal{L}_{kin} = \bar\psi i\hat\partial \psi$. Write $\psi = \psi_L + \psi_R$,

$\psi_{L,R} = \frac{1 \pm \gamma_5}{2}\psi$. $\bar\psi_{L,R} = \frac{\bar\psi(1 \mp \gamma_5)}{2} \Rightarrow$

$\mathcal{L}_{kin} = \bar\psi_L i\hat\partial\psi_L + \bar\psi_R i\hat\partial\psi_R$, so the left- and right-fields do not talk to each other. This is violated by the mass term $m\bar\psi\psi \equiv m(\bar\psi_L\psi_R + R \leftrightarrow L)$, but we will consider first the $m \to 0$ limit.

If $\psi_{L,R}$ decouple, we can perform an $SU(2)$ rotation on ψ_L & ψ_R separately. In other words

$\mathcal{L}_{kin}$ is invariant under:

$$\begin{cases} \psi_L \to \psi_L + \delta\psi_L, & \delta\psi_L = i\varepsilon_L^a T^a \psi_L \\ \psi_R \to \psi_R + \delta\psi_R, & \delta\psi_R = i\varepsilon_R^a T^a \psi_R \end{cases}$$

We say that the first transformation is generated by the group $SU(2)_L$, & the second - by the group $SU(2)_R$. The Lagrangian $\mathcal{L}_{kin}$ is invariant under $SU(2)_L \otimes SU(2)_R$.

It is instructive to write how the field ψ transforms under generic $SU(2)$ transformation:

$$\delta\psi = \delta\psi_L + \delta\psi_R = i\varepsilon_L^a T^a \psi_L + i\varepsilon_R^a T^a \psi_R =$$

$$= \left[i\varepsilon_L^a T^a \frac{(1+\gamma_5)}{2} + i\varepsilon_R^a T^a \frac{(1-\gamma_5)}{2} \right]\psi =$$

$$= i\left[\frac{(\varepsilon_L^a + \varepsilon_R^a)}{2} T^a + i\frac{(\varepsilon_L^a - \varepsilon_R^a)}{2}\gamma_5 T^a \right]\psi \Rightarrow$$

$$\delta\psi = i(\varepsilon^a - \gamma_5\varepsilon_5^a)T^a \cdot \psi, \quad \text{where}$$

$$\varepsilon^a = \frac{\varepsilon_R^a + \varepsilon_L^a}{2}, \quad \varepsilon_5^a = \frac{\varepsilon_R^a - \varepsilon_L^a}{2}.$$

For $\varepsilon^a \neq 0$, $\varepsilon_5^a = 0$, we have an isospin rotation, for $\varepsilon^a = 0$, $\varepsilon_5^a \neq 0$, we have a chiral rotation.

Of course, in the real world protons and neutrons aren't massless, so it is not clear why this limit is useful.

However, Gell-Mann and Levy realized that the masses can be given to proton & neutron as a consequence of spontaneous breaking of chiral symmetry.

To describe this, introduce a matrix field $\Sigma(x)$. We assume that Σ is a 2×2 matrix; it transforms under ~~SU(2)~~ $SU(2)_L \otimes SU(2)_R$ in the following way:

$$\Sigma \to L \Sigma R^+ \quad , R, L \in SU(2)_{R,L}$$

The Lagrangian is written as

$$\mathcal{L} = i\bar{\psi}\, i\hat{\partial}\psi - g\bar{\psi}_L \Sigma \psi_R - g\bar{\psi}_R \Sigma^+ \psi_L + \mathcal{L}(\Sigma)$$

Let us assume that $\mathcal{L}(\Sigma)$ is invariant under L, R transformations. This is easy to accomplish provided that

$$\mathcal{L}(\Sigma) \equiv f[\mathrm{Tr}(\Sigma\Sigma^+)]$$ where Tr is a trace in the $SU(2)$ space.

For the rest, a set of transformations

$$\psi_L \to \hat{L}\psi_L, \quad \psi_R \to \hat{R}\psi_R, \quad \Sigma \to \hat{L}\Sigma\hat{R}^+,$$

leave the Lagrangian invariant.

In general, an arbitrary 2×2 matrix can be written as a linear combination of an identity matrix and three pauli matrices with complex coefficients.

For our purposes we write ($\tau^a \equiv \sigma^a$, Pauli m.)

$$\boxed{\Sigma = \sigma + i\pi^a \cdot \tau^a}$$

, with σ & π^a, $a \in 1,2,3$ real. We will argue that the $L\Sigma R^+$ transformation keeps this form intact.

To this end, write

$$\Sigma\Sigma^+ = (\sigma + i\pi^a\tau^a)(\sigma - i\pi^b\tau^b) =$$

$$= \sigma^2 + \pi^a\pi^b\tau^a\tau^b = \sigma^2 + \pi^a\pi^b\delta^{ab} = (\sigma^2 + \vec{\pi}^2)$$

It follows that $\det\Sigma = \sigma^2 + \vec{\pi}^2$. It follows that $\Sigma = (\sigma^2 + \vec{\pi}^2)^{1/2} \cdot U$, where $\det U = 1$ and $U^+ U = 1$.

Now $\Sigma \to L\Sigma R^+ = (\sigma^2 + \vec{\pi}^2)^{1/2} L U R^+ = (\sigma^2 + \vec{\pi}^2)^{1/2} U_1$

$U_1 \in SU(2) \Leftarrow U_1^+ U_1 = 1$ and $\det U_1 = 1$ since both L & R are from $SU(2)$

To calculate transformation rules for (π^a, σ), we write

$$L \approx 1 + i\epsilon_L^a T^a$$
$$R \approx 1 + i\varepsilon_R^a T^a$$

$\Rightarrow$

$$\delta\Sigma = i\varepsilon_L^a T^a(\sigma + i\pi^b\tau^b) + (\sigma + i\pi^b\tau^b)(-i\varepsilon_R^a T^a)$$

$$= i\frac{(\varepsilon_L^a - \varepsilon_R^a)}{2}\tau^a\sigma - \frac{\pi^b\varepsilon_L^a}{2}\tau^a\tau^b + \frac{\tau^b\tau^a}{2}\varepsilon_R^a\pi^b$$

Now, $\tau^a\tau^b = \delta^{ab} + i\varepsilon^{abc}\tau^c \Rightarrow$

$$\frac{\pi^b\varepsilon_L^a}{2}\tau^a\tau^b = \frac{\vec{\pi}\cdot\vec{\varepsilon}_L}{2} + i\varepsilon^{abc}\frac{\varepsilon_L^a}{2}\pi^b\tau^c$$

$\Rightarrow$

$$\frac{\pi^b\varepsilon_R^a}{2}\tau^b\tau^a = \frac{\vec{\pi}\cdot\vec{\varepsilon}_R}{2} - i\varepsilon^{abc}\frac{\varepsilon_R^a}{2}\pi^b\tau^c$$

$$\delta\Sigma = -i\frac{(\varepsilon_R^a - \varepsilon_L^a)}{2}\tau^a\sigma + \frac{\vec{\pi}(\vec{\varepsilon}_R - \vec{\varepsilon}_a)}{2} - i\varepsilon^{abc}\frac{(\varepsilon_R^a + \varepsilon_L^a)}{2}\pi^b\tau^c$$

Since $\delta\Sigma = \delta\sigma + i\,\delta\pi^a\tau^a$, we find

$$\boxed{\delta\sigma = \vec{\pi}\cdot\vec{\varepsilon}_5 \qquad \delta\pi^a = -\,\varepsilon_5^a\cdot\sigma - \varepsilon^{abc}\varepsilon^b\pi^c}$$

We can use this decomposition of $\Sigma = \sigma + i\tau^a\pi^a$ to write

$$\mathcal{L} = i\bar{\psi}\, i\hat{\partial}\psi - g\bar{\psi}_L(\sigma + i\pi^a\tau^a)\psi_R - g\bar{\psi}_R(\sigma - i\pi^a\tau^a)\psi_L + \mathcal{L}(\Sigma) =$$

$$= i\bar{\psi}\, i\hat{\partial}\psi - g\bar{\psi}\psi\sigma - ig\pi^a(\bar{\psi}\gamma_5\tau^a\psi) + \mathcal{L}(\Sigma).$$

Our Lagrangian has $\bar{\psi}\psi\sigma$ an $\bar{\psi}\psi\pi^a$ coupling and is therefore reasonably close to what we want from phenomenology. The problem is that all particles are still massless. ~~Fi~~ To take care of that, we need $\mathcal{L}(\Sigma)$. To write it down, note that $\frac{1}{2}\mathrm{Tr}(\Sigma^+\Sigma) = \sigma^2 + \vec{\pi}^2$.

Similarly $\frac{1}{2}\mathrm{Tr}((\partial_\mu\Sigma)(\partial_\mu\Sigma^+)) = (\partial_\mu\sigma)(\partial^\mu\sigma) + (\partial_\mu\pi^a)^2$.

Hence, we can write

$$\mathcal{L}(\Sigma) = \frac{1}{4}\mathrm{Tr}\left((\partial_\mu\Sigma)(\partial_\mu\Sigma^+)\right) - \frac{\lambda}{4}\left(\frac{1}{2}\mathrm{Tr}(\Sigma^+\Sigma) - F_\pi^2\right)^2$$

$$= \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) + \frac{1}{2}(\partial_\mu\pi^a)(\partial^\mu\pi^a) -$$

$$- \frac{\lambda}{4}\left(\sigma^2 + \vec{\pi}^2 - F_\pi^2\right)^2.$$

This is the Lagrangian that develops spontaneous symmetry breaking.

To this end, write $\sigma = F_\pi + \sigma'$, and rewrite $\mathcal{L}$ using new fields.

We find {

$$\mathcal{L} = i\bar{\psi}\hat{\partial}\psi - gF_\pi\bar{\psi}\psi - g\sigma'\bar{\psi}\psi$$
$$+ ig\pi_a\bar{\psi}\tau_a\gamma_5\psi + \frac{1}{2}\partial^\mu\sigma'\partial_\mu\sigma' + \frac{1}{2}\partial^\mu\pi_a\partial_\mu\pi_a$$
$$- \frac{\lambda}{4}\left(\sigma'^2 + \pi_a^2 + 2F_\pi\sigma'\right)^2.$$

Hence, the theory describes:

1) a massive doublet $\psi = \binom{p}{n}$, with the mass gF_π.
2) a triplet of massless pions π^a;
3) a scalar field σ' with the mass $m_{\sigma'} = \sqrt{2\lambda F_\pi^2}$.

The pions are obviously Goldstone bosons of the theory. Let us clarify what symmetry is broken by our choice of the vacuum state:

$$\langle\sigma\rangle \equiv F_\pi, \quad \langle\pi^a\rangle = 0,$$
$$\hat{\Sigma}_{vac} \equiv F_\pi \cdot \mathbb{1}.$$

Upon a general $SU(2)_L \otimes SU(2)_R$ transf., we find

$$\hat{\Sigma}_{vac} \longrightarrow -i\tau^a\varepsilon_5^a \cdot F_\pi + F_\pi\mathbb{1}.$$

Hence, if $\varepsilon_5^a \neq 0$, the vacuum changes and our choice of vacuum breaks the symmetry associated with chiral rotations. Each broken symmetry corresponds to a massless Goldstone boson in the spectrum. Hence, ~~the~~ the three Goldstone bosons - the pions.

Let us calculate the Nöether current associated with the chiral symetry.

It reads

$$j_{5\mu}^{a} = -(\partial^{\mu}\pi_a)\sigma + (\partial^{\mu}\sigma)\pi^{a} - \bar{\Psi}\gamma^{\mu}\gamma_5\tau_a\Psi \quad *$$

Let's write this current in terms of "physical fields"; we have

$$j_{5\mu}^{a} = -(\partial^{\mu}\pi_a)F_{\pi} + \ldots$$

where ellipses stand for terms that are quadratic in the fields.

The first - linear - term is unusual. It is responsible for a matrix element between a Goldstone boson and the vacuum

$$\langle \pi^{a}_{(p)} | J_{5\mu}^{b} | 0 \rangle = i F_{\pi} p_{\mu} \delta^{ab}$$

The fact that the conserved current acts on a vacuum and produces a Goldstone boson is particular to theories with ~~electroweak~~ spontaneous symmetry breaking. The situation is very general. Here is a proof of the Goldstone theorem

To this end, consider a global continuous symmetry of the Lagrangian. Construct a Noether current $J^\mu(x)$. The current is conserved $\partial_\mu J^\mu(x) = 0$.
($J^{\dagger\mu} = J^\mu$, i.e. Hermitian current)

Conserved charge $Q = \int d^3\vec{x}\, J^0(\vec{x})$, $\dot{Q} = 0$

Define an order parameter
$\chi(x) = [Q, \phi(x)]$ where $\phi(x)$ is (eventually) an interpolating field for Goldstones

Consider a case when $\langle vac | \chi(x) | vac \rangle \neq 0$

This implies $Q|vac\rangle \neq 0$ and this implies that vacuum is not invariant under a symmetry transformation associated with $\hat{Q}$.

not an anti-commut. just a product

Consider $\Pi^\mu(q) = -i \int e^{iqx} d^4x \langle vac | T\{ J^\mu(x), \phi(0)\} | vac\rangle$

and

$$\lim_{q\to 0} q_\mu \Pi^\mu(q) = -i \int_{q\to 0} e^{iqx} d^4x\, q_\mu \langle vac | T\{ J^\mu(x), \phi(0)\} | vac\rangle$$
$$= - \int_{q\to 0} \frac{\partial}{\partial x_\mu} e^{iqx} d^4x \langle vac | T\{ J^\mu(x), \phi(0)\} | vac\rangle$$
$$= + \int_{q\to 0} e^{iqx} d^4x \frac{\partial}{\partial x_\mu} \langle vac | T\{ J^\mu(x), \phi(0)\} | vac\rangle$$

Now $\frac{\partial}{\partial x_\mu} \langle vac | T\{ J^\mu(x), \phi(0)\} | vac\rangle$

$$\equiv \langle vac | \delta(x_0) J^0(x) \phi(0) - \delta(x_0) \phi(0) J^0(x) | vac\rangle$$
$$= \delta(x_0) \langle vac | [J^0(x), \phi(0)] | vac\rangle \Rightarrow$$

$$\Rightarrow \lim_{q\to 0} q_\mu \Pi^\mu(q) = \int d^3\vec{x} \, \langle vac| [J^0(\vec{x}), \phi(0)] |vac\rangle$$
$$\equiv \langle vac| X |vac\rangle = v \neq 0$$

$\Rightarrow \quad \Pi^\mu(q) = \dfrac{v q^\mu}{q^2}$. Hence there is a q^2-pole in this correlator. As the result, there should be a massless particle that couples to both J^μ & φ.

It is instructive to explore various ways of representing massless degrees of freedom in the σ-model. To this end, we will only explore the mesonic sector of the theory

We have $\mathcal{L}(\Sigma) = \frac{1}{4} \operatorname{Tr}\left((\partial_\mu \Sigma)(\partial_\mu \Sigma^+)\right)$
$$- \frac{\lambda}{4}\left(\frac{1}{2}\operatorname{Tr}(\Sigma\Sigma^+) - F_\pi^2\right)^2$$

Writing $\Sigma = \sigma + i\pi^a \tau^a$ we obtain $\mathcal{L}(\sigma, \pi)$ with non-trivial coupling between σ & π.

Let us, however, write $\Sigma = \rho \times \mathcal{U}$, where $\mathcal{U} \in SU(2)$. Then $\Sigma\Sigma^+ = \rho^2 \mathcal{U}\mathcal{U}^+ = \rho^2 \mathbb{1}$

$$\partial_\mu \Sigma = (\partial_\mu \rho)\mathcal{U} + \rho\, \partial_\mu \mathcal{U}$$
$$\partial_\mu \Sigma^+ = (\partial_\mu \rho)\mathcal{U}^+ + \rho\, \partial_\mu \mathcal{U}^+ \quad \Rightarrow$$

$$\partial_\mu \Sigma\, \partial_\mu \Sigma^+ = (\partial_\mu \rho)(\partial^\mu \rho)\, \mathbb{1} + \rho(\partial_\mu \mathcal{U})\, \partial_\mu \rho\, \mathcal{U}^+$$
$$+ (\partial_\mu \rho)\, \mathcal{U}\, \rho\, \partial_\mu \mathcal{U}^+ + \rho^2 \partial_\mu \mathcal{U} \partial^\mu \mathcal{U}^+ \Rightarrow$$

Since $\partial_\mu(\mathcal{U}\mathcal{U}^+) = 0$,

$$\partial_\mu \Sigma\, \partial^\mu \Sigma^+ = \partial_\mu \rho\, \partial^\mu \rho\, \hat{\mathbb{1}} + \rho^2\, \partial_\mu \mathcal{U}\, \partial^\mu \mathcal{U}^+$$

$$\mathcal{L}(\Sigma) = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{\rho^2}{4} Tr(\partial_\mu U \partial^\mu U^\dagger)$$
$$- \frac{\lambda}{4} ~~Tr~~ (\rho^2 - F_\pi^2)^2$$

The symmetry breaking implies $\rho = F_\pi + \xi$

$$\rho^2 - F_\pi^2 = 2F_\pi \xi + \xi^2 \quad \partial_\mu \rho = \partial_\mu \xi \quad \text{and}$$

$$\mathcal{L}(\Sigma) = \frac{1}{2}(\partial_\mu \xi)^2 + \frac{1}{4}(F_\pi + \xi)^2 Tr(\partial_\mu U \partial^\mu U^\dagger)$$
$$- \frac{2\lambda F_\pi^2}{2} \xi^2 \left(1 + \frac{\xi}{2F_\pi}\right)^2. \quad (*)$$

Since $U = e^{\frac{i\pi^a(x)\tau^a}{F_\pi}}$, we have 3 massless pions and 1 massive field, exactly the same as when we represented Σ as $\sigma + i\pi^a\tau^a$.

For small values of the fields

$$F_\pi + \sigma' + i\pi^a\tau^a \approx (F_\pi + \xi) e^{i\frac{\tilde{\pi}^a\tau^a}{F_\pi}} \simeq (F_\pi + \xi)\left(1 + \frac{i\tilde{\pi}^a\tau^a}{F_\pi}\right)$$
$$\simeq F_\pi + \xi + \frac{i\tilde{\pi}^a\tau^a}{F_\pi} \Rightarrow$$

$\sigma' \simeq \xi$ & $\pi^a \simeq \tilde{\pi}^a$. But these equations are valid only for small field values.

Note that the Lagrangian $(*)$ is invariant under $U \to L U R^\dagger$ transformation.

The first point we make is that the Lagrangian $(*)$ and the Lagrangian $\mathcal{L}(\sigma, \vec{\pi})$ describe the same physics (same matrix elements for physics processes). Consider the limit $\lambda F_\pi^2 \gg E^2$, i.e. the field

ξ is heavy and can not get excited. Then it can be removed from the theory ("integrated out"). The result of this should be a Lagrangian that only depends on the massless fields : $U = e^{i\pi^a\tau^a/F_\pi}$ and that is invariant under chiral transformations $U \to L U R^+$. We will discuss how to construct such Lagrangian in the next lecture.