Lecture 2

Uniqueness of Yang-Mills theory

Interestingly, it is possible to argue that the Yang-Mills theory is <u>unique, in a certain sense</u>. To understand this, we will need to know a little bit about spinor-helicity formalism for massless quarks & gluons.

The key points are: the Dirac spinor for a fermion with momentum $\hat{p}_\mu$, $p^2=0$ is split into left & right ~~p~~ using projection operators

$$\hat{P}_{L,R} = \frac{1 \mp \gamma_5}{2} \qquad u(p) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix}$$

$$u_L(p) = \hat{P}_L u(p) \qquad u_R(p) = \hat{P}_R u(p).$$

For those spinors we use the notation

$$u_L(p) = p] \qquad u_R(p) = p\rangle$$

$$\overline{u_L(p)} = \langle p \qquad \overline{u_R(p)} = [p$$

The important points are

$$\langle p q] = [q p\rangle = 0, \quad \langle p p\rangle = [p p] = 0.$$

$$\langle p q\rangle = -\langle q p\rangle \qquad [p q] = -[q p]$$

$\hat{p} = |p\rangle [p| + |p]\langle p|$; The spinors $|p]$ & $|p\rangle$ give us the wave-functions for polarized fermions (quarks).

Gluons are spin-1 particles, so their wave functions are vectors $\varepsilon^{\mu}(p)$. The polarization vectors (outgoing gluons, skipping the complex conjugation):

$$\varepsilon_R^{\mu} = \frac{\langle r \gamma^{\mu} p]}{\sqrt{2}\langle r p\rangle}; \qquad \varepsilon_L^{\mu} = -\frac{[r \gamma^{\mu} p\rangle}{\sqrt{2}[r p]};$$

Here r^{μ} is the reference (gauge) vector. Nothing should depend on r^{μ}, which can be choosen independently for any external gluon ~~vector~~.

The scattering amplitudes describe interactions of on-shell particles. The scattering amplitudes can be written in terms of scalar products of polarization vectors and momenta or/and in terms of spinor products.

There is a particular symmetry transformation that tree amplitudes should satisfy when written in terms of spinor products.

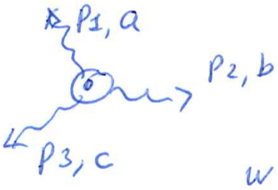
Since $\hat{p} = |p\rangle[p| + |p]\langle p|$, we can rescale $|p\rangle \to z|p\rangle$ & $[p| \to \frac{1}{z}[p|$, to keep $\hat{p}$ invariant. Of course $\langle p| \to z\langle p|$ and $|p] \to \frac{1}{z}|p]$ as well.

We'll be mostly concerned with gluon scattering amplitudes. Those depend on gluons polarization vectors that can be written in terms of spinors as shown above.

Upon z-rescaling, we find

$\varepsilon_R^{\mu} \to z^{-2}\varepsilon_R^{\mu}$ and $\varepsilon_L^{\mu} \to z^{2}\varepsilon_L^{\mu}$.

Now, let us attempt to write down an expression for the <u>3-gluon amplitude</u>.



The first thing to note is the momentum conservation, that we write as $\hat{p}_1 + \hat{p}_2 + \hat{p}_3 = \emptyset$.

Now, take $\langle 1|(\hat{p}_1 + \hat{p}_2 + \hat{p}_3) = 0$ and use the fact that $\hat{p}_i = |i\rangle[i| + |i]\langle i|$ and $\langle j i\rangle \neq 0$ but $\langle j i] = 0.$ $\Rightarrow$

$$\langle 1|\hat{p}_1 + \hat{p}_2 + \hat{p}_3 = \langle 12\rangle[2| + \langle 13\rangle[3| = 0$$
$$\langle 2|\hat{p}_1 + \hat{p}_2 + \hat{p}_3 = \langle 21\rangle[1| + \langle 23\rangle[3| = 0$$

Use $\langle 12\rangle = -\langle 21\rangle$, to write these equations as

$$\left.\begin{aligned} \langle 12\rangle[2| &= -\langle 13\rangle[3| \\ \langle 12\rangle[1| &= \langle 23\rangle[3| \end{aligned}\right\}$$

It follows from these equations that either $\langle 12\rangle = 0$, in which case $\langle 13\rangle = \langle 23\rangle = 0$ or $[2| \parallel [3| \parallel [1|$, in which case

$$[12] = [23] = [31] = 0.$$

For <u>real momenta</u> $\langle ij\rangle \doteq [ji]^*$ and, therefore, if $\langle ij\rangle = 0 \Rightarrow [ij] = 0$. Hence, <u>it is impossible</u> to construct non-trivial amplitude for <u>real momenta</u>; <u>all spinor products vanish</u>. However, this is not true <u>for complex momenta</u>.

So, let's imagine that $p_{1..3}$ are complex.
Then, either $\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0$ or
$[12] = [23] = [31] = 0$. Therefore, amplitudes can either be functions of $\{\langle 12\rangle, \langle 23\rangle, \langle 31\rangle\}$ or of $\{[12], [23], [31]\}$.

Consider first the scattering amplitude $M(1^a_L, 2^b_L, 3^c_L)$
The amplitude should depend on 3 polarization vectors, each should scale as Z^2. Let's write

$$M(1^a_L, 2^b_L, 3^c_L) \equiv C^{abc} \langle 12\rangle^{n_1} \langle 23\rangle^{n_2} \langle 31\rangle^{n_3}$$

Rescaling $|1\rangle$ with z_1, $|2\rangle$ with z_2, $|3\rangle$ with z_3, we obtain 3 equations

$$\begin{cases} n_1 + n_3 = 2 \\ n_1 + n_2 = 2 \\ n_2 + n_3 = 2 \end{cases} \Rightarrow n_1 = n_2 = n_3 \equiv 1.$$

$$\Rightarrow M(1^a_L, 2^b_L, 3^c_L) = C^{abc} \langle 12\rangle\langle 23\rangle\langle 31\rangle$$

If, on the other hand, we assume that <u>this</u> amplitude depends on $\{[ij]\}$, we'll find

$$M(1^a_L, 2^b_L, 3^c_L) = C^{abc} \frac{1}{[12][23][31]}.$$

Since for real momenta $[ij] \to 0$, the second expression for the amplitude has an ∞ limit, instead of zero and, therefore, is not acceptable.

Hence, we conclude

$$M(1^a_L, 2^b_L, 3^c_L) = C^{abc} \langle 12\rangle\langle 23\rangle\langle 31\rangle$$

Next, since $\langle ij\rangle[ji] = 2p_i \cdot p_j$, the ~~dim~~ mass dimension of $\langle ij\rangle$ and $[ij]$ is one.

Since the mass dimension of the gluon scattering amplitude is "μ", we either have to have $C^{abc} \sim \frac{1}{\mu^2}$, in which case the theory is not renormalizable or we need to conclude that $C^{abc} \equiv \phi$. Hence,

$$M(1_L^a, 2_L^b, 3_L^c) \equiv \phi.$$

Similar arguments apply to $M(1_R^a, 2_R^b, 3_R^c) = \phi$.

Next, consider $M(1_L^a, 2_L^b, 3_R^c)$. Assume

$$M(1_L^a, 2_L^b, 3_R^c) = C^{abc} \langle 12\rangle^{n_1} \langle 23\rangle^{n_2} \langle 31\rangle^{n_3}$$

$M(1_L^a, 2_L^b, 3_R^c)$ should scale as $z_1^2 \cdot z_2^2 \, z_3^{-2}$.

$$\Rightarrow \quad n_1 + n_3 = 2 \qquad n_1 + n_2 = 2 \qquad n_2 + n_3 = -2$$

$$\Rightarrow \quad n_1 + n_2 + 2n_3 = 0 \quad \Rightarrow \quad 2 + 2n_3 = 0 \quad \Rightarrow \quad n_3 = -1, \quad n_2 = -1, \quad n_1 = 3$$

$$\Rightarrow \quad M(1_L^a, 2_L^b; 3_R^c) = C^{abc} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

The second option is

$$M(1_L^a, 2_L^b, 3_R^c) = \bar{C}^{abc} \cancel{\frac{[31]}{[12][23]}} \cdot \frac{[23][31]}{[12]^3}$$

In the limit of real momenta, $\frac{[23][31]}{[12]^3} \to \infty$ and, therefore, not acceptable. ~~while $\frac{[31]}{[12][23]}$ ...~~ ~~Therefore,~~ The acceptable form of the 3-point scattering amplitude is

$$M(1_L^a, 2_L^b; 3_R^c) = C^{abc} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

Similarly,

$$M(1_R^a, 2_R^b, 3_L^c) = C^{abc} \frac{[12]^3}{[23][31]}$$

Finally, we note that we have just calculated **bosonic** amplitude, that should be symmetric. But the amplitudes that we have written is antisymmetric e.g. with respect to $1 \leftrightarrow 2$ permutations. This must be compensated by the antisymmetry of C^{abc} that should be totally antisymmetric.

Next, let us consider the 4-gluon amplitude. $M(1_L^a, 2_L^b, 3_R^c, 4_R^d)$. Here, the kinematics is already much less restrictive than in the 3-gluon case and real momenta satisfying $\sum_{i=1}^{4} p_i = 0$ are possible. So, we write a general expression of the amplitude consistent with the "z-scaling":

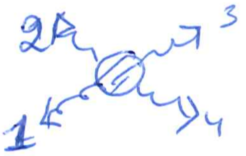
$$M(1_L^a, 2_L^b, 3_R^c, 4_R^d) = \langle 12\rangle^2 [34]^2 F^{abcd}(s,t,u),$$

where $s = (p_1+p_2)^2$, $t = (p_1+p_3)^2$, $u = (p_1+p_4)^2$.

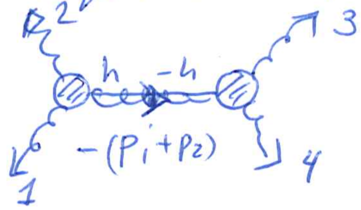
As usual $s+t+u=0$. We note that the 4-point amplitude has the mass dimension zero which means that $F^{abc}(s,t,u) \sim \frac{1}{\mu^4}$.

General consequence of the unitarity of the scattering matrix is that amplitudes should factorize on kinematic poles in a particular way, corresponding to exchanges of on-shell intermediate states.

Let us consider the pole in the s-channel. Then, the following equation should hold:



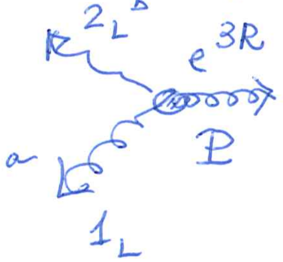
$$\Bigg|_{(p_1+p_2)^2 \to 0} \longrightarrow \sum_h$$



Let us understand restrictions on the function $F^{abc}(s,t,u)$ that such factorization should provide. First, let us define $P = -p_1 - p_2 = p_3 + p_4$

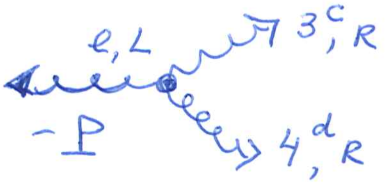
The propagator of the "off-shell" gluon is $\dfrac{i\delta^{ab}}{P^2}$ and $1/P^2$ provides the required pole..

Now we use our results for ~~this~~ the 3-point helicity amplitudes. We have, for the left vertex



$$\equiv C^{abe} \frac{\langle 12\rangle^3}{\langle 2P\rangle\langle P1\rangle}$$ is the nonvanishing amplitude.

while for the right vertex



$$\equiv C^{cde} \frac{[34]^3}{[3\,{-P}][{-P}\,4]}$$

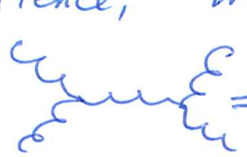
Spinors for momentum $-P$ can be related to spinors with P in a simple way

$|-P] = i|P]$ $\quad |-P\rangle = i|P\rangle$ $\quad \langle -P| = i\langle P|$ $\quad [P| = i[P|$

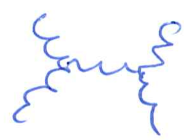
(to motivate this, note $|P]\langle P| + |P\rangle [P| = \hat{P}$, so that $-\hat{P} = |-P]\langle -P| + \ldots = i^2 |P]\langle P| + \ldots$
$= i^2 \hat{P} = -\hat{P}$).

Hence, we obtain :

$s \to 0$



$$= -\frac{i C^{abe} C^{cde}}{s} \frac{\langle 12\rangle^3 [34]^3}{\langle 2P\rangle\langle P1\rangle [3P][P4]}$$

Defining  $\equiv iM$, we obtain

$$\lim_{s\to 0} s\, M(1_L^a, 2_L^b, 3_R^c, 4_R^d) = +C^{abe}C^{cde}\frac{\langle 12\rangle^3[34]^3}{\langle 2P\rangle\langle P1\rangle[3P][4P]}$$

We can simplify this expression using the momentum conservation. Since $P=-1-2$,

$$\langle 2P\rangle[P| = -\langle 21\rangle[1| \Rightarrow \langle 2P\rangle[P4] = -\langle 21\rangle[14]$$

$$\langle 1P\rangle[P| = -\langle 12\rangle[2| \Rightarrow \langle 1P\rangle[P3] = -\langle 12\rangle[23]$$

$$\Rightarrow \langle 2P\rangle\langle P1\rangle[3P][4P] = +\langle 21\rangle[14](-1)\langle 12\rangle[23]$$
$$= \langle 12\rangle^2[14][23].$$

Now $\langle 12\rangle[23] = \langle 1|\hat{2}|3] = \langle 1|-\hat{4}|3] = \langle 14\rangle[34]$.

$$\Rightarrow \langle 2P\rangle\langle P1\rangle[3P][4P] = \langle 12\rangle[34][14]\langle 14\rangle \Rightarrow$$

$$\lim_{s\to 0} s\, M(1_L^a, 2_L^b, 3_R^c, 4_R^d) = +C^{abe}C^{cde}\frac{\langle 12\rangle^2[34]^2}{\langle 14\rangle[14]}$$

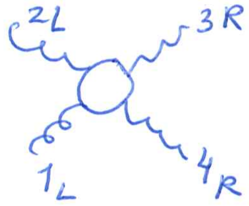
We go back to our original definition of the 4-point scattering amplitude in terms of the function $\mathcal{F}^{abcd}(s,t,u)$ (page 6) and find

$$\lim_{s\to 0} s\,\mathcal{F}^{abcd}(s,t,u) = +\frac{C^{abe}C^{cde}}{-\langle 14\rangle[41]} = +\frac{C^{abe}C^{cde}}{-s_{14}}$$

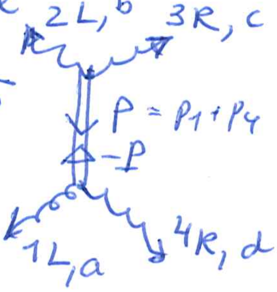
Define $(p_1+p_4)^2 = s_{14} = t$, we obtain

$$\boxed{\lim_{s\to 0} st\,\mathcal{F}^{abcd}(s,t,u) \equiv -C^{abe}C^{cde}}$$

Next, we can consider a factorization in the t-channel

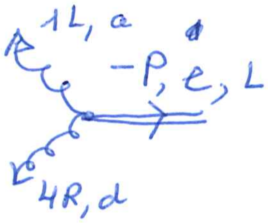


$$\xrightarrow[t\to 0]{} \sum_h$$

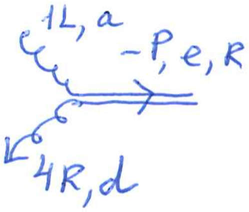


$$t = (p_1+p_4)^2 = (p_2+p_3)^2$$

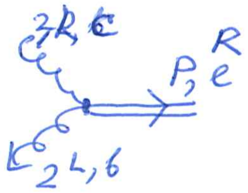
Now, let's write down possible 3-point amplitudes.



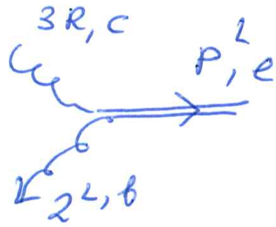
$$= C^{aed} \frac{\langle 1\,-P\rangle^3}{\langle -P\,4\rangle\langle 41\rangle} = - C^{aed}\frac{\langle 1P\rangle^3}{\langle P4\rangle\langle 41\rangle}$$



$$= C^{eda}\frac{[-P\,4]^3}{[41][1\,-P]} = - C^{eda}\frac{[P4]^3}{[41][1P]}$$



$$\equiv C^{ceb}\frac{[3P]^3}{[P2][23]}$$



$$\equiv C^{ebc}\frac{\langle P2\rangle^3}{\langle 23\rangle\langle 3P\rangle}$$

We now combine the two possible contributions to obtain

$$\lim_{t\to 0} iM^{abcd}(s,t,u) = \frac{i}{t}\left\{ -C^{aed}\frac{\langle 1P\rangle^3}{\langle P4\rangle\langle 41\rangle}\, C^{ceb}\frac{[3P]^3}{[P2][23]} \right.$$

$$\left. \equiv C^{eda}\frac{[P4]^3}{[41][1P]}\, C^{ebc}\frac{\langle P2\rangle^3}{\langle 23\rangle\langle 3P\rangle} \right\} =$$

$$= \frac{i}{t} C^{ade} C^{bce} \left\{ \frac{\langle 1P\rangle^3}{\langle 14\rangle\langle 4P\rangle} \frac{[3P]^3}{[32][2P]} + \frac{[4P]^3}{[41][1P]} \frac{\langle 2P\rangle^3}{\langle 23\rangle\langle 3P\rangle} \right\}$$

Now, we simplify it again using momentum conservation. $P = p_1 + p_4 = -p_2 - p_3 \Rightarrow$

$$\langle 1P\rangle [P3] = \langle 1|\hat{P}|3] = \langle 14\rangle [43]$$

$$[4P]\langle P2\rangle = [4|\hat{P}|2\rangle = [41]\langle 12\rangle$$

The rest becomes

$$\langle 4P\rangle [2P] = -\langle 4P\rangle [P2] = -\langle 4|\hat{P}|2] = -\langle 4|\hat{1}+\hat{4}|2]$$
$$= -\langle 41\rangle [12] = \langle 14\rangle [12]$$

$$[1P]\langle 3P\rangle = -[1P]\langle P3\rangle = [14]\langle 34\rangle \quad \Rightarrow$$

$$\lim_{t\to 0} t M^{abcd} = C^{ade} C^{bce} \left\{ \frac{-\langle 14\rangle^3 [43]^3}{\langle 14\rangle [32] \langle 14\rangle [12]} \right.$$
$$\left. + \frac{[41]^3 \langle 12\rangle^3}{[41]\langle 23\rangle [14]\langle 34\rangle} \right\} = C^{ade} C^{bce} \left\{ \frac{\langle 41\rangle [34]^3}{[32][21]} \right.$$
$$\left. + \frac{[14]\langle 21\rangle^3}{\langle 23\rangle\langle 34\rangle} \right\}.$$

Now, since $t = (p_1+p_4)^2 = [14]\langle 41\rangle$, we can approach the limit $t\to 0$ in the complex plane by either taking $[14]\to 0$ or $\langle 41\rangle \to 0$. The above expression shows that – in principle – the results of the calculation can be different.

In practice, they are the same (how?), so take the ~~second~~ first one, i.e. $[14] \to 0$ will be used. This leaves us with:

$$\lim_{t\to 0} t M^{abcd} = C^{ade} C^{bce} \frac{\langle 41\rangle [34]^3}{[32][21]}$$

Now, $$[34]^3\langle 41\rangle = [34]^2 [34]\langle 41\rangle = [34]^2 [3\hat{4}1\rangle =$$
$$= -[34]^2 [32]\langle 21\rangle = \langle 12\rangle^2 [34]^2 \left(-\frac{[32]}{\langle 12\rangle}\right)$$

Next

$$\lim_{t\to 0} t M^{abcd} = C^{ade} C^{bce} \frac{\langle 12\rangle^2 [34]^2 [32]}{[32][21]\langle 12\rangle} \approx$$

$$= \mp C^{ade} C^{bce} \frac{\langle 12\rangle^2 [34]^2}{\langle 12\rangle [21]} \Rightarrow$$

$$\boxed{\lim_{t\to 0} st\, \mathcal{F}^{abcd}(s,t,u) = \mp C^{ade} C^{bce}}$$

With the u-channel the story is similar and so we get:

$$\boxed{\lim_{u\to 0} su\, \mathcal{F}^{abcd}(s,t,u) = C^{ace} C^{bde}}$$

Having derived the unitarity constraints, we can now ask the question if the function $\mathcal{F}^{abcd}(s,t,u)$ actually exists.

What do we know about F^{abcd}?

First, we know that its mass dimension is -4.
Second, since $u=-s-t$, it can be thought of as a function of one dimensionless variable. For convinience, we write

$$F^{abcd}(s,t,u) = \frac{1}{st} f_1^{abcd}\left(\frac{s}{t}\right) + \frac{1}{tu} f_2^{abcd}\left(\frac{u}{t}\right)$$

Next, we study the limits $s\to 0$, $t\to 0$, $u\to 0$ and require that ~~only~~ simple poles in all these variables are the strongest singularities that can appear.

Suppose we write $f_1^{abcd}\left(\frac{s}{t}\right) = \sum_n a_n^{1,abcd}\left(\frac{s}{t}\right)^n$

$$f_2^{abcd}\left(\frac{u}{t}\right) = \sum_n a_n^{2,abcd}\left(\frac{u}{t}\right)^n$$

From the absence of s & u poles, the above sums should go from $n=0$ to $n=\infty$

From the s pole, we have

$$a_0^{1,abcd} = -C^{abe}C^{cde}$$

From the u pole, we have

$$a_0^{2,abcd} = C^{ace}C^{bde}$$

To get the information from the t-pole, ~~the situation~~ is more difficult.

We find $(t\to 0,\ u\to -s)$

$$\boxed{a_0^{1,abcd} - a_0^{2,abcd} \equiv C^{ade}C^{bce}}$$

and relations between $a_{n>0}^{1,abcd}$ and $a_{n>0}^{2,abcd}$ that ensure that a single pole in t is the strongest singularity. The last equation for a_0 coefficients implies:

$$\boxed{C^{abe} C^{cde} + C^{ace} C^{bde} + C^{ade} C^{bce} \equiv 0}$$

From the 3-point amplitudes, we know that C^{abc} is totally anti-symmetric. Hence, the above equation can be interpreted as the Jacobi identity of a Lie algebra.

Hence, we conclude that gauge theories based on Lie algebras are unique consistent interacting theories ~~of~~ with massless spin-1 fields.