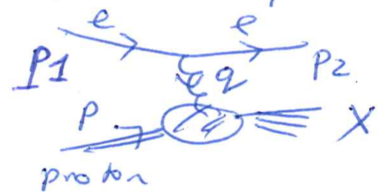
Deep inelastic scattering

Consider the process $e + p \to e + X$ (electron-proton scattering). We are not interested in the fate of a proton after the collision.

Pictorially :



The matrix element $M = -ie^2 \bar{u}(p_2)\gamma^\mu u(p_1) \frac{-i}{q^2}\langle H|J_\mu|N\rangle$

We then write the differential cross section in a standard way

$$d\sigma_H = \frac{1}{4\mathcal{J}} \times \underbrace{\frac{1}{2}\Big|_e \times \frac{1}{2}\Big|_N}_{spin} \frac{(4\pi\alpha)^2}{(q^2)^2} L^{\mu\nu} \sum_{spin_N} |\langle N|J_\mu|H\rangle|^2 \times$$

$$\times (dp_2)(dH)(2\pi)^4 \delta(p_1 + p - p_2 - p_H)$$

Here $\mathcal{J} = p_1 \cdot p$ is the flux & $(dp_2)(dH)$ is the phase space volume element. Also $q = p_1 - p_2$, $L^{\mu\nu} = Tr(\hat{p}_2 \gamma_\mu \hat{p}_1 \gamma_\nu)$. We now sum over all possible Hadrons in the final state

$$d\sigma = \sum_H d\sigma_H = \frac{(4\pi\alpha)^2}{4sq^4} \int d^4q\, \delta(p_1 - p_2 - q) \times$$

$$\times L_{\mu\nu} \frac{1}{2} \sum_{spin_N} \sum_H (dp_2)(dH)(2\pi)^4 \delta(q + p - p_H) \langle N|J_\mu|H\rangle\langle H|J_\nu|N\rangle$$

$$(dp_2) \equiv \frac{d^4p_2\, 2\pi \delta_+(p_2^2)}{(2\pi)^4} \quad \& \quad \delta(q + p - p_H) = \int d^4x\, e^{i(q+p-p_H)\cdot x}$$

$$p_2 = (q - p_1)^2$$

$\Rightarrow$

$$d\sigma = \frac{(4\pi\alpha)^2}{4sq^4} \frac{d^4q}{(2\pi)^4} 2\pi \delta_+(q^2 - 2qp_1) \times L_{\mu\nu} \times \frac{1}{2} \sum_{spin_N}$$

$$\int d^4x\, e^{iqx} \sum_H (dH) \langle N|J_\mu(x)|H\rangle\langle H|J_\nu(0)|N\rangle =$$

We can now write:

$$\sum_H (dH) \langle N | J_\mu(x) | H \rangle \langle H | J_\nu(0) | N \rangle \equiv$$

$$\equiv \langle N | J_\mu(x) \, J_\nu(0) | N \rangle \quad \Rightarrow$$

$$\boxed{d\sigma = \frac{(4\pi\alpha)^2}{4 s q^4} \frac{d^4 q}{(2\pi)^4} \, 2\pi \, \delta(q^2 - 2 q p_1) \times L_{\mu\nu} \times \frac{1}{2} \sum_{\substack{\text{spin} \\ N}} \int d^4 x \, e^{iqx} \langle N | J_\mu(x) \, J_\nu(0) | N \rangle.}$$

We introduce $W_{\mu\nu} \equiv \frac{1}{4\pi} \frac{1}{2} \sum_{\substack{\text{spin} \\ N}} \int d^4 x \, e^{iqx} \langle N | J_\mu(x) J_\nu(0) | N \rangle$

$$\Rightarrow \boxed{d\sigma = \frac{2\alpha^2}{s (q^2)^2} L_{\mu\nu} \cdot W^{\mu\nu} \, d^4 q \, \delta(q^2 - 2 p_1 \cdot q)}$$

Next, we need to parametrize the hadronic tensor $W^{\mu\nu}$. Since $\partial_\mu J^\mu = 0$, $q_\mu W^{\mu\nu} = \phi$. Since $W_{\mu\nu}$ depends on 2 vectors, p_μ & q_μ, ~~$W_{\mu\nu}$ is~~ and since ~~the~~ parity is conserved, we can write

$$\boxed{W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1(q^2, pq) + \left(p_\mu - \frac{pq}{q^2} q_\mu\right)\left(p_\nu - \frac{pq}{q^2} q_\nu\right) W_2}$$

The lepton tensor $L_{\mu\nu}$ is also conserved ($L_{\mu\nu} \cdot q_\mu = 0$). Therefore

$$L_{\mu\nu} W^{\mu\nu} = -g_{\mu\nu} L^{\mu\nu} W_1 + p_\mu p_\nu L^{\mu\nu} W_2$$

$$L^{\mu\nu} = 4 \left(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu} p_1 \cdot p_2\right) \Rightarrow$$

$$L_{\mu\nu} g^{\mu\nu} = -8 p_1 \cdot p_2 \; ; \quad p_\mu p_\nu L^{\mu\nu} = 8 (p \cdot p_2)(p \cdot p_1),$$

since $p^2 = \phi$.

The scalar products can be expressed in terms of $p_1 \cdot p_2 = -q^2/2 \quad p_2 \cdot p = (p_1 - q)\,p = \frac{s}{2}\left(1 - \frac{2pq}{s}\right)$, where $s = p_1 \cdot p_2$.

For the total cross-section, we find

$$d\sigma = \frac{2\alpha^2}{s(q^2)^2}\, d^4q\ \delta(q^2 - 2p_1 q)\left[-4q^2 W_1 + 2s^2\left(1 - \frac{2pq}{s}\right) W_2\right]$$

Now, we can integrate over q. To this end:

$q = \xi p + \beta p_1 + q_\perp$: $d^4q = \frac{s}{2}\, d\xi\, d\beta\, d\vec{q}_\perp^{\,2}$

$q^2 = s\xi\beta - \vec{q}_\perp^{\,2}$. Introduce $Q^2 = -q^2$ &

write $d\vec{q}_\perp^{\,2} \equiv dQ^2$.

Also $\delta(q^2 - 2p_1 q) = \delta(s\xi\beta - \vec{q}_\perp^{\,2} - 2\xi s) = \delta(Q^2 + s\xi)$

Integrating over ξ, we find:

$$\frac{d\sigma}{dQ^2 d\beta} = \frac{4\pi\alpha^2}{sQ^4}\left(Q^2 W_1 + \frac{s^2}{2}\left(1 - \frac{2pq}{s}\right) W_2\right)$$

Conventional variables for DIS are

$$\boxed{x = \frac{Q^2}{2pq} \quad \text{and/or} \quad y = \frac{2pq}{s}} \Rightarrow$$

$y = \frac{2pq}{s} = \beta \qquad x = \frac{Q^2}{s\beta} \Rightarrow d\beta = \frac{Q^2 dx}{sx^2} \Rightarrow$

$$\frac{d\sigma}{dQ^2 dx} = \frac{d\sigma}{dQ^2 d\beta}\,\frac{Q^2}{sx^2} = \frac{4\pi\alpha^2}{s^2x^2Q^2}\left(Q^2 W_1 + \frac{s^2}{2}(1-y) W_2\right)$$

Using $xy = \frac{Q^2}{s}$ and $\frac{Q^2}{s^2x^2} = \frac{y^2}{Q^2} \quad \frac{y^2 s^2}{2Q^2} = \frac{pq}{x}$,

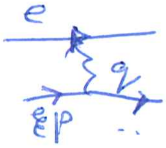
we can write the cross-section in the canonical form:

$$\frac{d\sigma}{dQ^2 dx} = \frac{4\pi\alpha^2}{Q^4}\left\{ y^2 W_1 + \frac{1}{x}(1-y)(pq) W_2 \right\}$$

Often, instead of W_1 & W_2 two other functions – F_1 & F_2 are used. They are defined $\boxed{W_1 = F_1 \quad \text{and} \quad F_2 = (pq)\, W_2.}$

– So far we have discussed a general parametrization of deep inelastic scattering. The result is expressed in terms of 2 functions, F_1 and F_2. These functions, in general, depend on Q^2 and x. Let us calculate F_1 and F_2 in the parton model where it is assumed that the scattering of electron on proton occurs on proton constituents that carry momentum fraction ξ of the proton momentum, $p_c = \xi P$. These constituents can have different electric charges. The probability to find a constituent of the type i with momentum fraction ξ in a proton is given by $d\xi\, f_i(\xi)$, where $f_i(\xi)$ is called parton-distribution function. So, in the parton model

$$\boxed{d\sigma_{eP} = \sum_i \int d\xi\, f_i(\xi)\; d\sigma_i(e + \xi P \to e' + X)}$$

$d\sigma(e + \xi P \to e' + X)$ receives contribution from 

We assume that partons are spin 1/2 particles, whose interaction with photons is facilitated by normal vector currents.

Then $(p\xi \to p_q)$ (e_i is the electric charge)

$$d\sigma_i = \frac{e_i^2}{4 p_q \cdot p_1} \frac{1}{4} \frac{(4\pi\alpha)^2}{(q^2)^2} L^{\mu\nu} H_q^{\mu\nu} \cdot \frac{d^3\vec{p}_2}{(2\pi)^3 2E_{p_2}} \frac{d^3\vec{p}_q{}'}{(2\pi)^3 2E_{q'}}$$

$$\times (2\pi)^4 \delta^{(4)}(p_1 + p_q - p_2 - \vec{p}_q{}')$$

Now $H_q^{\mu\nu} = \mathrm{Tr}(\hat{p}_{q'} \gamma^\mu \hat{p}_q \gamma^\nu) = 4(p_{q'}^\mu p_q^\nu + p_{q'}^\nu p_q^\mu - g^{\mu\nu} p_{q'} \cdot p_q)$

Now, write

$$\frac{d^3\vec{p}_2}{(2\pi)^3 2E_{p_2}} \frac{d^3\vec{p}_{q'}}{(2\pi)^3 2E_{q'}} (2\pi)^4 \delta^{(4)}(p_1 + p_q - p_2 - p_{q'}) =$$

$$= \frac{d^4 p_2}{(2\pi)^4} 2\pi\delta(p_2^2) \frac{d^4 p_{q'}}{(2\pi)^4} 2\pi\delta(p_{q'}^2)(2\pi)^4 \delta^{(4)}(p_1 + p_q - p_2 - p_{q'})$$

$$= \frac{d^4 p_2}{(2\pi)^4} 2\pi\delta(p_2^2)\, 2\pi\delta\left((p_1 + p_q - p_2)^2\right) =$$

$$= \frac{d^4 q}{(2\pi)^4} 2\pi\delta\left((p_1 - q)^2\right) 2\pi\delta\left((q + p_q)^2\right)$$

Again $q = \alpha p_q + \beta p_1 + q_\perp$ $\quad d^4 q = \frac{s}{2} d\alpha\, d\beta\, d^2 q_\perp =$

$$= \frac{s\pi}{2} d\alpha\, d\beta\, d q_\perp^2$$

$-Q^2 = s\alpha\beta - \vec{q}_\perp^{\,2} \Rightarrow$

$d^4 q = \frac{s\pi}{2} d\alpha\, d\beta\, dQ^2 \qquad \delta\left((p_1 - q)^2\right) = \delta(-Q^2 + s\alpha)$

$$\frac{d\sigma}{dQ^2 d\beta} = \frac{\cancel{s}\pi}{2} \times \frac{e_i^2}{\cancel{s}} \frac{1}{2\cancel{s}} \times \frac{1}{4} \frac{(4\pi\alpha)^2}{(Q^2)^2} L_{\mu\nu} H_q^{\mu\nu}$$

$$\times \frac{1}{(2\pi)^4} (2\pi)^2 \delta(-Q^2 + \beta s) =$$

$$= \frac{16\pi^3 \alpha^2 e_i^2}{16 \cdot s\, \cancel{4}\pi^2 (Q^2)^2} L_{\mu\nu} H_q^{\mu\nu} \delta(\beta s - Q^2)$$

To compute $L_{\mu\nu} H_q^{\mu\nu}$, it is useful to do the following: First write $H_q^{\mu\nu}$ as

$$H_q^{\mu\nu} = 4\left(p_q^{\mu} p_{q'}^{\nu} + p_q^{\nu} p_{q'}^{\mu} - g^{\mu\nu} p_q \cdot p_{q'}\right).$$ Now,

use $p_q \cdot p_{q'} = -q^2/2$ and $p_{q'} = p_q + q$, to rewrite it as

$$H_q^{\mu\nu} = 4\left[\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\left(-\frac{q^2}{2}\right) + 2\left(p_q^\mu - q^\mu \frac{p_q \cdot q}{q^2}\right)\left(p_q^\nu - q^\nu \frac{p_q \cdot q}{q^2}\right)\right]$$

The two structures are identical to what appeared in the hadronic tensor in the "hadronic" calculation. ~~Then~~ To keep those structures identified, we'll keep tags on them.

$$H_q^{\mu\nu} = 4\left[\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\left(-\frac{q^2}{2}\right)W_1^q + 2\left(p_q^\mu - q^\mu \frac{p_q \cdot q}{q^2}\right)\left(p_q^\nu - q^\nu \frac{p_q \cdot q}{q^2}\right)W_2^q\right]$$

Using $L_{\mu\nu} g^{\mu\nu} = -8 p_1 p_2$ $\quad L_{\mu\nu} q_\mu = 0$ $\quad L_{\mu\nu} p_q^\mu p_q^\nu = 8(p_q p_2)(p_q p_1)$,

we write

$$\frac{d\sigma^q}{dQ^2 d\beta} = \frac{\pi \alpha^2 e_i^2}{4 s (Q^2)^2} \cdot 4\left[-4q^2\left(-\frac{q^2}{2}\right)W_1^q + 2s^2\left(1 - \frac{2pq}{s}\right) 2 W_2^q\right]\delta(\beta s - Q^2)$$

$$\frac{d\sigma^q}{dQ^2 d\beta} = \frac{\pi \alpha^2 e_i^2}{s (Q^2)^2}\left[2Q^4 W_1^q + 4s^2\left(1 - \frac{2pq}{s}\right)W_2^q\right]\delta(\beta s - Q^2)$$

Next: the quantity s here is "partonic", i.e. $2p_q \cdot p_1$

$p_q = \xi P$ & so $s = \xi s_h$, where $s_h = 2P \cdot p_1$ ("hadronic")

We write the above expression in hadronic variables

$$\frac{d\sigma_i}{dQ^2 d\beta} = \frac{\pi \alpha^2 e_i^2}{s_h \xi (Q^2)^2}\left[2Q^4 W_1^q + 4\xi^2 s_h^2\left(1 - \frac{2pq}{s_h}\right)W_2^q\right]\delta(\beta \xi s_h - Q^2)$$

We use again hadronic x: $x = \frac{Q^2}{s_h \beta}$, to write

$$\frac{d\sigma_i}{dQ^2 dx} = \frac{d\sigma_i}{dQ^2 d\beta}\frac{Q^2}{s_h x^2} = \frac{\pi \alpha^2 e_i^2 Q^2}{s_h^2 \xi x^2 (Q^2)^2}\left[2Q^4 W_1^q + 4\xi^2 s_h^2 (1-y) W_2^q\right] \times$$
$$\times\, \delta\left(\frac{Q^2 \xi}{x} - Q^2\right)$$

$$\frac{d\sigma_i}{dQ^2dx} = \frac{\pi\alpha^2 e_i^2}{(Q^2)^2}\frac{1}{S_h^2 x\xi}\left[2Q^4 W_1^q + 4\xi^2 S_h^2(1-y)W_2^q\right]\delta(\xi - x) =$$

$$= \frac{\pi\alpha^2 e_i^2}{(Q^2)^2}\left[\frac{2Q^4}{S_h^2 x^2}W_1^q + 4(1-y)W_2^q\right]\delta(\xi - x)$$

Now, the hadronic cross-section is

$$\frac{d\sigma}{dQ^2dx} = \sum_i \int d\xi\, f_i(\xi)\,\frac{d\sigma_i}{dQ^2dx} \equiv$$

$$\equiv \frac{4\pi\alpha^2}{Q^4}\sum_i \frac{e_i^2 f_i(x)}{4}\left[2y^2 W_1^q + \frac{4}{x}(1-y)\cdot x W_2^q\right]$$

Hence we find: $(W_1^q = 1,\ W_2^q = 1)$

$$\left\{\begin{aligned} F_1(Q^2,x) &= \frac{1}{2}\sum_i e_i^2 f_i(x) \\ F_2(Q^2,x) &\equiv \sum_i e_i^2 x f_i(x) \end{aligned}\right. \Rightarrow \boxed{F_1 \cdot 2x = F_2}$$

"Callan - Gross relation".

The above results show that structure functions – F_1 and F_2 – are independent of Q^2 in the parton model. This was observed experimentally in deep inelastic scattering experiments (scaling) ~1970. This observation proved that protons have point-like constituents – partons (or quarks etc.)

We will now try to understand how the parton model appears from QCD.

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To understand this, we will need to use the operator product expansion, so we go back to the representation of the hadronic tensor $W_{\mu\nu}$ as the matrix element of currents:

$$W_{\mu\nu} = \frac{1}{4\pi} \frac{1}{2} \sum_{\text{spin}_N} \int d^4y \, e^{iqy} \langle N | J_\mu(y) J_\nu(0) | N \rangle$$

To connect is to Green's functions, consider

$$T_{\mu\nu} = \frac{1}{4\pi^2} \frac{1}{2} \sum_{\text{spin } N} \int d^4y \, e^{iqy} \langle N | T \, J_\mu(y) J_\nu(0) | N \rangle .$$

Taking imaginary part of $T_{\mu\nu}$ gives $W_{\mu\nu}$:

$$W_{\mu\nu} = \mathrm{Im}(T_{\mu\nu})$$

Let us discuss the tensor $T_{\mu\nu}$ in the kinematics of deep inelastic scattering.

We ~~find~~ have $q^2 = -Q^2 < 0$ and $\frac{Q^2}{2pq} = x$.

We are interested in the limit $Q^2 \to \infty$, x – fixed. We would like to understand the values of y^μ ~~for~~ which contribute to the $T_{\mu\nu}$ integral in that limit.

To this end, write $\begin{cases} q = \alpha p + \beta n + q_\perp \\ y = a p + b n + y_\perp \end{cases}$,

where $p^2 = 0$ $n^2 = 0$ $p \cdot n \neq 0$ $q_\perp \cdot p = 0$ $q_\perp \cdot n = 0$

$y_\perp \cdot p = 0$ $y_\perp \cdot n = 0$

Then $q \cdot p = \beta \cdot pn \Rightarrow \beta = \frac{q \cdot p}{pn} = \frac{Q^2}{2x(pn)}$.

Take $p \cdot n = 1$, so that $\beta = \frac{Q^2}{2x}$.

$\alpha = q \cdot n$. Since $q^2 = 2\alpha\beta pn - \vec{q}_\perp^{\,2} \Rightarrow$

$-Q^2 = Q^2/x \cdot \alpha - \vec{q}_\perp^{\,2} \Rightarrow \alpha \simeq x \sim 1$ & $\vec{q}_\perp^{\,2} \sim Q^2$.

Since $qy \sim \alpha B + a\beta - \vec{q}_\perp \cdot \vec{y}_\perp$, ~~we find that~~ and since the contribution to the integral over y comes from the region where $qy \sim 1$, we find $y_\perp \sim \sqrt{1/Q^2}$ $\quad \alpha \sim \frac{1}{Q^2}$ $\quad B \sim 1$.

Hence, the component of y along "n^μ" is not restricted. On the other hand, since $y^2 \simeq 2\alpha B \mp \vec{y}_\perp^{\,2} \sim \frac{1}{Q^2} \to 0$, the major contribution to the final result comes from the light cone. Hence, in the DIS limit, the correlator of the currents $\langle N | T J_\mu(y) J_\nu(0) | N \rangle$ is probed on the light cone, $y^2 \to 0$, but the points ~~y~~ & y does not approach $y = 0$. ~~pointwise~~.

This observation has important consequences. Let us write, schematically, an OPE of 2 scalar currents: [$J = \bar{\Psi}(x)\Psi(x)$] (Scalar currents are used to avoid ~~Lore~~ having to do with Lorentz indices)

$$T\,J(y)\,J(0) \simeq \sum_{ij} C_i^j(y^2)\, y^{\mu_1} \cdots y^{\mu_j}\, \mathcal{O}^{ij}_{\mu_1 \cdots \mu_j}$$

The two indices i & j refer to the following things: "i" describes a type of operator & "j" describes different "spins" of otherwise the same operators.

For example, an operator $\bar{\Psi}\Psi$ & the operator $\bar{\Psi} D_\mu \Psi$ differ by "spin", while $\bar{\Psi}\sigma^{\mu\nu}\Psi$ differ by "type".

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We'll now take the matrix element of $TJ(y)J(0)$ w.r.t. nucleon states and average over spins

$$\frac{1}{2}\sum_{spin} \langle N|TJ(y)J(0)|N\rangle = \sum_{i,j} C_i^j(y^2)\, y^{\mu_1}\cdots y^{\mu_j} \langle N|\mathcal{O}^{ij}_{\mu_1\ldots\mu_j}|N\rangle$$

$\langle N|\mathcal{O}^{ij}_{\mu_1\ldots\mu_j}|N\rangle$ is a rank-j tensor that is composed out of momentum of the nucleon p_μ and the metric tensor $g_{\mu\nu}$; the polarization vector isn't available because of spin average.

Then

$$\langle N|\mathcal{O}^{ij}_{\mu_1\ldots\mu_j}|N\rangle = \mathcal{O}^{ij}(p^2)\,[p_{\mu_1}\ldots p_{\mu_j}] + \text{terms with } g^{\mu\nu}.$$

Terms with $g^{\mu\nu}$ lead to $y^2 \sim 1/q^2 \to 0$, as $q^2\to\infty$; therefore, they can be dropped.

We obtain

$$T(q^2) = \frac{1}{4\pi^2}\int d^4y\, e^{iqy} \sum_{ij} C_i^j(y^2)(yp)^j\,\mathcal{O}^{i,j},$$ where

$\{\mathcal{O}^{i,j} = \mathcal{O}^{i,j}(p^2)\}$ are regular numbers, not operators. Note, that it can be obtained from a generic OPE expressions by computing

$$\mathcal{O}^{ij} = \frac{1}{2}\sum_{\substack{spin\\N}} \langle N|n_{\mu_1}\ldots n_{\mu_j}\,\mathcal{O}^{i,j\,\mu_1\ldots\mu_j}|N\rangle \qquad \begin{pmatrix} p\cdot n = 1\\ n^2 = 0\end{pmatrix}$$

As the next step, we need to integrate over y:

$$\int d^4y\, e^{iqy}\, C_i^j(y^2)\,(yp)^j \equiv$$

$$\equiv p^{\mu_1}\ldots p^{\mu_j}\int d^4y\, e^{iqy}\, C_i^j(y^2)\, y_{\mu_1}\ldots y_{\mu_j} =$$

$$= p^{\mu_1}\ldots p^{\mu_j}\,\frac{\partial}{i\partial q_{\mu_1}}\cdots\frac{\partial}{i\partial q_{\mu_j}}\int d^4y\, e^{iqy}\, C_i^j(y^2)$$

$$= p^{\mu_1} \ldots p^{\mu_j} \frac{\partial}{i\partial q_{\mu_1}} \cdots \frac{\partial}{i\partial q_{\mu_j}} C_i^j(q^2) .$$

Now, $\displaystyle \frac{\partial}{\partial q_\mu} C(q^2) = \frac{\partial C}{\partial q^2} 2q^\mu$

$$\frac{\partial}{\partial q_\nu}\frac{\partial}{\partial q_\mu} C(q^2) = \frac{\partial^2 C}{\partial^2 q^2} 2q^\nu 2q^\mu + \frac{\partial C}{\partial q^2} 2g^{\mu\nu}$$

The last term gets contracted with $p^\mu \ldots p^\nu g_{\mu\nu} \to \phi$ and can be dropped. Hence, we find

$$\int d^4y\, e^{iqy} C_i^j(y^2)(yp)^j \equiv \frac{(2pq)^j}{i^j}\left(\frac{\partial}{\partial q^2}\right)^j C_i^j(q^2) =$$

$$= \left(\frac{2pq}{-q^2}\right)^j (iq^2)^j \left(\frac{\partial}{\partial q^2}\right)^j C_i^j(q^2)$$

Hence, we find

$$T(q^2, x) = \frac{1}{4\pi^2} \sum_{ij} \mathcal{O}^{ij} x^{-j} (iq^2)^j \left(\frac{\partial}{\partial q^2}\right)^j C_i^j(q^2)$$

$$= \frac{1}{4\pi^2} \sum_j x^{-j} \sum_i \mathcal{O}^{ij}\, \bar{C}_i^j(q^2) ,$$

where the dependence ~~of~~ on x & q^2 are separated.

To describe deep-inelastic scattering, we need $W_{\mu\nu}$, not $T_{\mu\nu}$. We know that $W_{\mu\nu}$ is the imaginary part of $T_{\mu\nu}$.

To obtain the full function from its imaginary part, we use dispersion relations ~~for~~ in the variable $\omega = pq$, at fixed q^2.

Schematically, $\displaystyle T(q^2, \omega) = \int d\omega' \frac{W(q^2, \omega')}{\omega' - \omega}$

The integration region is somewhat tricky: $T(q^2,\omega)$ develops imaginary part in two cases: $\omega < -Q^2/2$ & $\omega > \frac{Q^2}{2}$. Since $x = Q^2/2pq = Q^2/2\omega$, the region where T has imaginary part is $-1 < x < 1$.

Write $\omega = \frac{Q^2}{2x}$ $\quad d\omega = \frac{Q^2}{2x^2}\, dx \quad \Rightarrow$

$$T(q^2,x) = \int_{-1}^{1} \frac{dx'}{x'^2} \frac{W(q^2,x')}{1/x' - 1/x} = \int_{-1}^{1} \frac{dx'}{x'} \frac{x W(q^2,x')}{x - x'}$$

To make contact with the OPE expression, consider $x > 1$. Then $\frac{1}{x-x'}$ can be expanded in x'/x. We find

$$T(q^2,x) = \sum_j x^{-j} \int_{-1}^{1} dx'\, W(q^2,x')\, x'^{\,j-1}$$

Comparing with the OPE result, we find

$$\boxed{\int dx\, x^{j-1} W(q^2,x) = \sum_i O^{ij} C_i^j(Q^2)}$$

As the next step, we need to understand what are the operators that contribute to the sum.

In the standard OPE, contributions of operators with higher mass dimensions are suppressed. In case of the ~~D~~DIS, the situation is different.

The relevant parameter is called "twist", the difference between ~~the~~ mass dimension of an operator and its spin.
We can see this from the discussion at the beginning: for fixed "i" (type) mass dimension increases with "j" (spin), but as we see from the previous formula, terms with higher "j" are not $1/Q^2$ suppressed. The reason is that $y \cdot P$ is not small.

In QCD, there are 3 types of operators of ~~spin~~ twist 2:

1) $O_q^{\mu_1 \ldots \mu_n} = \frac{i^{n-1}}{n!} \{ \bar{q}(x) \gamma^{\mu_1} D^{\mu_2} \ldots D^{\mu_n} q(x) + \text{permutation of indices} \}$

2) $O_{q_1 q_2}^{\mu_1 \ldots \mu_n}$ – a non-singlet generalization of $O_q^{\mu_1 \ldots \mu_n}$

3) $O_g^{\mu_1 \ldots \mu_n} = \frac{i^{n-2}}{n!} \operatorname{tr} \left[G^{\mu_1 \mu_2} D^{\mu_2} \ldots D^{\mu_{n-1}} G^{\mu_{n-1} \mu_n} \right]$

Even if we restrict ourselves to $O_q^{\mu_1 \ldots \mu_n}$, there is infinitely large number of ~~them~~ operators that we have to consider! To understand what we can do, we focus on the contribution of O_q operators. Their Wilson coefficients

at leading order in perturbation $C_i^j(Q^2)$
theory are constants. Then, from the moment equation for $W(q^2,x)$, it follows that

$$(*) \qquad \int dx\, x^{j-1}\, W(q^2,x) = \langle N|\bar{q}\,\hat{n}\,(i n_\mu D^\mu)^j q|N\rangle$$

Choose the light-cone gauge $n_\mu \hat{A}^\mu = 0$.
Then $n_\mu D^\mu = n_\mu \partial^\mu$. To simplify $(*)$, we multiply both sides of the equation with $(-i\lambda)^j/j!$ and take the sum over j.

$$\sum_{j=0}^{\infty}\int dx\, x^{j-1}\, W(q^2,x)\,\frac{(-i\lambda)^j}{j!} = \sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\langle N|\bar{q}\,\hat{n}\,(i\partial_n)^j q|N\rangle$$

$$\Longrightarrow \int_0^1 \frac{dx}{x}\, W(q^2,x)\, e^{-i\lambda x} = \langle N|\bar{q}(0)\,\hat{n}\,q(\lambda n)|N\rangle$$

$$\Rightarrow \quad \frac{W(q^2,x)}{x} = \int \frac{d\lambda}{2\pi}\, e^{i\lambda x}\, \langle N|\bar{q}(0)\,\hat{n}\,q(\lambda n)|N\rangle$$

This quantity $\frac{W(q^2,x)}{x}$ is called the parton distribution function:

$$f_q(x) = \int \frac{d\lambda}{2\pi}\, e^{i\lambda x}\, \langle N|\bar{q}(0)\,\hat{n}\,q(\lambda n)|N\rangle.$$

The above calculation was done assuming that $C^j(Q^2) = \text{const}$. If this is not the case, we can rewrite the Eq. on page 12 $\int dx\, x^{j-1} W(q^2,x) = \sum_i O^{ij}\cdot C_i^j(Q^2)$

in a useful way. Namely, the ~~LHC~~ l. h. s. can be considered as a Mellin transform. Then, there is a mathem. statement that a mellin transform of a convolution is a product of mellin transforms. Hence,

$$d\sigma(x, Q^2) \sim W(Q^2; x) = \int dz\, d\xi\, \delta(x - z\xi)\, f_q(\xi) \times \sigma_{q+e \to X}(z, Q^2)$$

This is the ~~ke~~ sketch of how the major parton model equation is derived from QCD.