QCD Lagrangian, general properties asymptotic freedom

Physics of strong interactions is described by a gauge theory known as QCD (Quantum Chromodynamics). The gauge group is $SU(3)$.

The Lagrangian of QCD is

$$\mathcal{L} = \mathcal{L}_{classic} + \mathcal{L}_{gauge-fix} + \mathcal{L}_{ghosts}$$

The Lagrangian $\mathcal{L}_{classic}$ is

$$\mathcal{L}_{classic} = -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} + \sum_{i \in fl} \bar{q}_i (i\hat{D} - m_i) q_i ,$$

where $\hat{D} = D_\mu \gamma^\mu$, $D_\mu = \partial_\mu + ig T^a A^a_\mu$ and

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu .$$

Fields A^a_μ are <u>gluons</u> (spin-1), fields q_i are Dirac fermions (quarks). Quarks may have different flavors (u, d, s, c, t, B), characterized by the mass m_i. Also, T^a, $a \in 1..8$, are generators of $SU(3)$.

The Lagrangian $\mathcal{L}_{classic}$ is gauge-invariant.

This means that if we change

$$q(x) \to q'(x) = \Omega(x) q(x) , \quad \Omega(x) = e^{iT^a \theta^a(x)} ,$$

$$\bar{q}(x) \to \bar{q}'(x) = \bar{q}(x) \Omega^+(x) = \bar{q}(x) e^{-iT^a \theta^a(x)}$$

$$\hat{A}(x) \to \hat{A}'(x) = \Omega \hat{A} \Omega^{-1} + \frac{i}{g} [\partial_\mu \Omega] \Omega^{-1} ,$$

the Lagrangian $\mathcal{L}_{classic}$ does not change.

Useful relations to prove the invariance of $\mathcal{L}_{classic}$ are $D_\mu \to D'_\mu = \Omega D_\mu \Omega^{-1}$ and $\hat{G}_{\mu\nu} = G^a_{\mu\nu} T^a \to \hat{G}'_{\mu\nu} = \Omega G_{\mu\nu} \Omega^{-1}$.

An important consequence of gauge symmetry is that if $A \equiv M_\mu \varepsilon^\mu(k)$ is the scattering amplitude that involves the gluon with momentum k, $M_\mu \cdot k^\mu = \phi$. Of course, A should be physical amplitude, which means that all particles there should be on the mass shell.

$\mathcal{L}_{gauge-fix}$ is the gauge-fixing Lagrangian; it involves the condition that the gauge field should satisfy. For example, $\mathcal{L}_{gauge-fix} = -\frac{1}{2\lambda}(\partial_\mu A^a_\mu)^2$, for a class of covariant gauges. With this gauge fixing term, the gluon propagator ~~reads~~ satisfy the following equation

$$(k^2 g_{\mu\nu} - (1-\lambda^{-1}) k_\mu k_\nu) \mathcal{D}_{\nu\rho} = -i g_{\nu\rho}$$

Solving it, we find $\mathcal{D}_{\mu\nu} = \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\lambda)\frac{k_\mu k_\nu}{k^2}\right)$.

The ghost term $\mathcal{L}_{ghost}$ reads

$\mathcal{L}_{ghost} = \partial_\mu \bar{c}^a D^{ab}_\mu c^b$; ghosts are scalars with Fermi-Dirac statistic.

They are needed to cancel unphysical contributions of the gluons fields, when improper sums of polarization are used for the latter.

Note that there are examples of gauge conditions where ghosts fields are not needed; these are the so-called physical gauges (axial, Coulomb). In ~~these~~ the axial gauges, the condition $\mathcal{L}_{gauge-fix} = -\frac{1}{2\lambda}(n_\nu A^\mu)^2$, where $n_\mu^2 = 0$. The limit $\lambda \to 0$, $n^2 \to 0$ implies that the gluon propagator reads

$$D_{\mu\nu} = \frac{-i}{k^2+i0}\left(g_{\mu\nu} - \frac{1}{k\cdot n}\left(k_\mu n_\nu + k_\nu n_\mu\right)\right).$$

The residue of the gluon propagator at $k^2=0$ gives polarization sum for the gluon

$$\lim_{k^2\to 0}(-ik^2 D_{\mu\nu}) = -g_{\mu\nu} + \frac{1}{k\cdot n}\left(k_\mu n_\nu + k_\nu n_\mu\right) \equiv \sum_{\lambda=\mp 1} \varepsilon_\mu \varepsilon^*_\nu$$

The polarization sum is transverse - physical and satisfy $\boxed{\varepsilon_\lambda \cdot k = 0,\ \varepsilon_\lambda \cdot n = 0}$

The QCD Lagrangian posesses a remarkable property which is called asymptotic freedom. The coupling constant g depends on the scale and satisfy an equation ($\alpha_s = g^2/4\pi$)

$$\mu \frac{dg}{d\mu} = -\beta(g)\cdot g \ , \quad \beta(g) = +\beta_0 \cdot \frac{\alpha_s}{4\pi} + O(\alpha_s^2)$$

$$\beta_0 = \frac{11}{3}N - \frac{2}{3}n_f \ ,$$ where $N=3$ is the number of colors and n_f is the number of fermions spieces.

This equation implies that

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\beta_0}{2\pi}\alpha_s(\mu_0)\ln\frac{\mu}{\mu_0}}, \text{ and so } \mu \to \infty,$$

$\alpha_s(\mu) \to 0$. At the same time, since $\alpha_s(\mu)$ can be written as $\alpha_s(\mu) = \frac{1}{\frac{\beta_0}{2\pi}\ln\frac{\mu}{\Lambda_{QCD}}}$, there is a scale $\mu \sim \Lambda_{QCD} \sim 300$ MeV, where the QCD coupling constant becomes very large.

So, QCD at large energies (small distances) is <u>"perturbative"</u>; QCD at <u>small energies</u> (large distances) is <u>"non-perturbative."</u>

The <u>decrease</u> of the coupling constant at short distances is counter-intuitive; it is violating the wisdom that we have from the QED calculations where effect of the charge screening implies that the QED coupling at short distances <u>increases</u> & the QED coupling at large distances decreases. <u>It is interesting to understand where this difference is coming from.</u>

The simplest way to do that is to use QCD quantized in the Coulomb gauge, i.e. $\partial_i A_i(x) = 0$. The propagator in this gauge splits into a propagator for <u>Coulomb</u> gluons and for the <u>transverse</u> gluons:

$$D^{ab,00}(q_0,\vec{q}) = +i\delta^{ab} D_c(q_0,\vec{q}) \qquad D_c(q_0,\vec{q}) = \frac{+1}{\vec{q}^{\,2}}$$

$$D^{ab,ij}(q_0,\vec{q}) = i\delta^{ab} D^{ij}(q_0,\vec{q}) \qquad D_{ij}(q_0,\vec{q}) = \frac{1}{q^2+i0}\left(\delta^{ij} - \frac{\vec{q}^{\,i}\vec{q}^{\,j}}{\vec{q}^{\,2}}\right)$$

The thing to notice is that the propagator of the Coulomb gluon is <u>independent of q_0</u>; this will be important for the future.

To calculate the β-function, we will study the interaction of very heavy quarks with each other. The quarks are non-relativistic. The Coulomb gluons couple to $\bar{u}\gamma^0 u$; the transverse gluons to $\bar{u}\gamma^i u$

In the static limit, $u(m,\vec{0}) = \sqrt{2m}\begin{pmatrix}\varphi\\0\end{pmatrix}$

$$\gamma^0 = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \Rightarrow \bar{u}\gamma^0 u = u^\dagger u = 2m(\varphi^\dagger\cdot\varphi)$$

But $\bar{u}\gamma^i u = 2m(\varphi^\dagger,\vec{0})\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}\begin{pmatrix}\varphi\\0\end{pmatrix} \equiv 0.$

We conclude, therefore, that static quarks do not interact with transverse gluons and only <u>Coulomb gluons are relevant</u>.

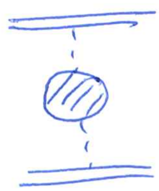
Next, we calculate the scattering amplitude and relate it to the Fourier transform of the interaction potential. For color-singlet $q\bar{q}$ pair, we find (g_0 is the bare coupling)

$$\mathcal{U}(\vec{q}) = -C_F g_0^2 D_c(q_0,\vec{q})$$

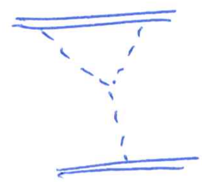
The one-particle irreducible correction to the Coulomb propagator are given by $-i\vec{q}^{\,2}\,\Pi(q^2,\vec{q}^{\,2})$; the full Coulomb propagator becomes

$$\boxed{D^{ab,00}_{full} = +i\delta^{ab}\,\frac{D_C(q_0,\vec{q})}{1-\Pi(q^2,\vec{q}^{\,2})}}$$

. Therefore, the modification of the interaction potential is caused by $\Pi(q^2,\vec{q}^{\,2})$ that we will have to calculate.

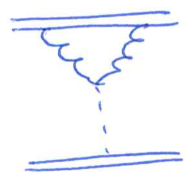


Note that no other corrections are possible since a vertex with 3 Coulomb gluons is absent (antisymmetry), so



isn't possible

and



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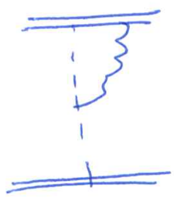
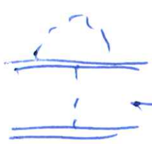


vanishes in the static limit

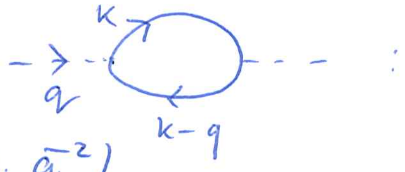


$\to \emptyset$

Hence, the Coulomb gauge gives us an opportunity to compute corrections to QCD interaction potential by considering corrections to vacuum polarization of the Coulomb gluon.

There are 3 diagrams that contribute

We will start with a simple case – ~~the~~ the quark loop. To simplify the calculation, we will use <u>dispersion representation</u> of the loop diagram



:

$$\Pi(q^2, \bar{q}^2) = \int_0^\infty ds \, \frac{\rho(s, \bar{q}^2)}{q^2 - s + i0}$$

Calculate the discontinuity

$$\mathrm{Disc}\left(-i\bar{q}^2 \Pi(q^2, \bar{q}^2)\right) = -i\bar{q}^2(-2\pi i)\,\rho(q^2, \bar{q}^2) = -2\pi \bar{q}^2 \rho(q^2, \bar{q}^2)$$

The same discontinuity is obtained from the ~~same~~ Feynman integral by taking $\frac{1}{p^2 + i0} \to -2\pi i\, \delta(p^2)$. We find $\left(v^\mu = (1, \vec{0})\right)$

$$\mathrm{Disc}\left(-i\bar{q}^2 \Pi(q^2, \bar{q}^2)\right) = (-1)\, T_R\, g_0^2 \int \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left(\hat{v}\hat{k}\hat{v}(\hat{k} - \hat{q})\right) \times (-2\pi i)^2 \delta(k^2)\, \delta((k-q)^2)$$

$$\equiv 4\pi^2 g_0^2 T_R \int \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left(\hat{v}\hat{k}\hat{v}(\hat{k} - \hat{q})\right) \delta(k^2)\, \delta((k-q)^2)$$

We will only need the result of the calculation for $q_0^2 \gg \bar{q}^2$ or $q^2 \gg \bar{q}^2$ since we are trying to understand logarithmic renormalization of couplings.

First, let us understand the constraints that $\delta(k^2)$ and $\delta((k-q)^2)$ imply for the momentum ~~conservation~~ integration:

$k^2 = 0 \Rightarrow k_0 = \pm|\vec{k}|$. But for $q_0 > 0$, $k_0 > 0$ as well and so $k_0 = +|\vec{k}|$.

Then:

$$0=(q-k)^2=q^2-2q\cdot k=q_0^2-\vec{q}^{\,2}-2q_0k_0+2k_0|\vec{q}|\cos\theta$$

Write $k_0=\frac{q_0}{2}(1+\delta)$ ~~and~~ with $\delta\ll 1$. Then solve the above constraint iteratively.

We find ($r=|\vec{q}|/q_0\ll 1$)

$$k_0=|\vec{k}|=\frac{q_0}{2}\left(1+r\cos\theta-r^2\sin^2\theta)+O(r^3)\right)$$

Next, computing the trace, we find

$$Tr(\hat{v}\hat{k}\hat{v}(\hat{k}-\hat{q}))=4\left(2(k\cdot v)(v(k-q))-k\cdot(k-q)\right)=$$

$$=4\left(2k_0(k_0-q_0)+k\cdot q\right)=4\left(2k_0^2-2k_0\cdot q_0+q^2/2\right)$$

$$=2q_0^2(\delta^2-r^2)=-2q_0^2r^2\left(1-(\vec{n}_k\cdot\vec{n}_q)^2\right)$$

Now, we can easily complete calculation of the discontinuity

First $\int\frac{d^4k}{(2\pi)^4}\delta(k^2)\delta((q-k)^2)\underset{r\to 0}{\cong}\int\frac{d\Omega_{\vec{k}}}{8(2\pi)^4}$. So that

$$-2\pi\vec{q}^{\,2}\rho(q^2,\vec{q}^{\,2})=T_R\int\frac{d\Omega_{\vec{k}}}{8(2\pi)^4}\,4\pi^2g_0^2\left(-2q_0^2r^2\right)\left(1-(\vec{n}_k\cdot\vec{n}_q)^2\right)$$

Hence,

$$\rho(q^2,0)=2T_R\frac{g^2}{(4\pi)^2}\int\frac{d\Omega_{\vec{k}}}{4\pi}\left(1-(\vec{n}_k\cdot\vec{n}_q)^2\right)\Rightarrow$$

$$\boxed{\rho(q^2,0)=T_R\frac{g^2}{(4\pi)^2}\cdot\frac{4}{3}}$$

With this result, we can calculate the change in the interaction potential between the quarks. The relevant point is $q_0\ll|\vec{q}|$ & $q^2=-\vec{q}^{\,2}$

$$\mathcal{U}(\vec q) = -\frac{C_F g_0^2 D_c(q_0,\vec q)}{1-\Pi(-\vec q^2,\vec q^2)} = -\frac{C_F g_0^2}{\vec q^2}\left(1+\Pi(-\vec q^2,\vec q^2)\right)$$

$$= -\frac{C_F g_0^2}{\vec q^2}\left(1-\int_{\vec q^2}^{\Lambda^2}\frac{ds}{s}\rho(s)\right) = -\frac{C_F g_0^2}{\vec q^2}\left(1+\frac{T_R g_0^2}{(4\pi)^2}\frac{4}{3}\ln\frac{\vec q^2}{\Lambda^2}\right)$$

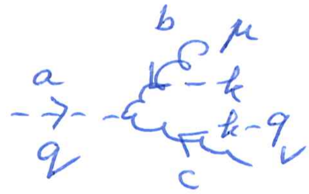
We can interpret this result as the "running coupling constant", i.e. writing

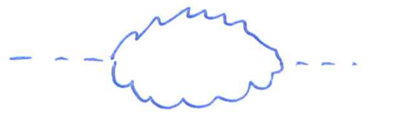
$\mathcal{U}(\vec q) = -\dfrac{C_F\, g^2(\vec q^2)}{\vec q^2}$, to obtain

$$g^2(\vec q^2) = g_0^2\left(1+\frac{T_R g_0^2}{(4\pi)^2}\frac{4}{3}\ln\frac{\vec q^2}{\Lambda^2}\right)$$ which can be recognized as a relation between the couplings at a scale Λ^2 and $\vec q^2$.

The sign reflects the "screening", familiar from Abelian theories.

Next, we consider a contribution were the Coulomb ~~gauge~~ gluon splits into two transversal gluons. The 3-gluon vertex for a Coulomb gluon (a, q) splitting into two transverse (all momenta incoming) reads:

$$-g_s f^{abc}(q-2k)_0\, g_{\mu\nu} \equiv$$ 

We write the discontinuity for the diagram with two intermediate transversal gluons  :

Disc [] = $\mathrm{Disc}\left[\frac{g^2}{2} f^{abc} f^{dcb} \int \frac{d^4k}{(2\pi)^4} \times\right.$

$$\left. \times\, (q_0 - 2k_0)^2 \; \frac{-i\,P_{ij}(k)}{k^2 + i0} \; \frac{-i\,P_{ij}(k-q)}{(k-q)^2 + i0} \right] = \left(P_{ij} = \delta_{ij} - n_i n_j \right)$$

$$= \frac{g^2}{2} f^{abc} f^{dcb} \left(-4\pi^2\right) \int \frac{d^4k}{(2\pi)^4} (q_0 - 2k_0)^2 \cdot$$

$$\cdot\, P_{ij}(k) \cdot P_{ij}(k-q)\; \delta(k^2)\, \delta((k-q)^2) .$$

The "kinematics" of particles in the loop was already discussed in the calculation of quark loop. There we found that $q_0 - 2k_0 = -|\bar{q}| \cos\theta$. The integrand is already proportional to $(q_0 - 2k_0)^2 \equiv \bar{q}^2 \cos^2\theta$, so we do not need any other terms in $\bar{q}$, in particular $P_{ij}(k)\, P_{ij}(k+q) \approx P_{ij}(k)\, P_{ij}(k) \equiv 2$.

The spectral density is then easy to calculate and one obtains $\rho_T = \frac{g_s^2 C_A}{(4\pi)^2} \frac{1}{3}$

Comparing this with the fermion contribution, we find that the contribution of the transverse gluon is obtained from the contribution of quark by changing

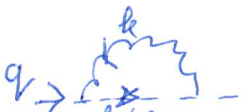
$$\frac{4}{3} T_R \to \frac{C_A}{3} .$$

Hence, the full change in the coupling is

$$g^2(\bar{q}^2) = g_0^2 \left(1 + \frac{g_0^2}{(4\pi)^2} \left(T_R \cdot \frac{4}{3} N_f + \frac{C_A}{3} \right) \ln \frac{\bar{q}^2}{\Lambda^2} \right),$$

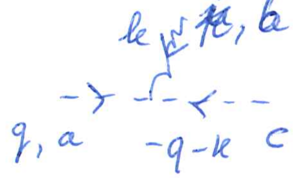
so that -- again -- there effect is screening, the same as in the Abelian theory

The last diagram we have to consider is



and this diagram differs from everything that we have seen so far. The reason is that ~~[illegible]~~ the Coulomb gluon propagator does not depend on q_0 and, therefore, it doesn't have discontinuity in q^2.

We will do the calculation directly. The vertex for 2 gluons Coulomb and 1 transverse is



$$= -g f^{abc}(-2q-k)_i \,.$$ Then

$$= g^2 C_A \delta^{ab} \int \frac{d^4 k}{(2\pi)^d} \frac{(2q+k)_i (2q+k)_j P_{ij}(k)}{(k^2+i0)\left[(\vec{k}+\vec{q})^2\right]}$$

Since $P_{ij}(k)\cdot k_i = 0$, the numerator simplifies. The remaining integration is ~~done~~ performed by doing k_0 integration using residue theorem. Picking up the pole at $k_0 = -|\vec{k}|$, we obtain

$$= \frac{-i g^2 C_A \delta^{ab}}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{4 q_i q_j P_{ij}(\vec{k})}{|\vec{k}|(\vec{k}+\vec{q})^2} =$$

We are interested in logarithmic divergence of this integral. We have then

$$= \frac{-i g^2 C_A \delta^{ab}}{2} \frac{\vec{q}^{\,2}}{3} 4 \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\delta^{ij} P_{ij}(\vec{k})}{|\vec{k}|^3} =$$

$$= -\frac{i g^2 C_A \delta^{ab}}{2} 4 \frac{\vec{q}^{\,2}}{3} \frac{4\pi}{(2\pi)^3} \frac{2}{2} \ln \frac{\Lambda^2}{\vec{q}^{\,2}} = -i \vec{q}^{\,2} \frac{g^2}{(4\pi)^2} C_A \delta^{ab} \frac{16}{3} \ln \frac{\Lambda^2}{\vec{q}^{\,2}}$$

This result <u>has a different sign</u> compared to two previous contributions. It, therefore, implies an anti-screening. Unfortunately, our quantization procedure was too cavalier, so the coefficient that we obtained is not correct.

The correct result $\frac{16}{3} \to 4$. [The full calculation in the Coulomb gauge is described in T. D. Lee "Particle Physics and Introduction to Field Theory".]

With the above modification, we obtain

$$g^2(\bar{q}^2) = g_0^2 \left\{ 1 + \frac{g_0^2}{(4\pi)^2} \left[-4C_A + \frac{C_A}{3} + \frac{4}{3} T_R N_f \right] \ln \frac{\bar{q}^2}{\Lambda^2} \right\}$$

The term in square brackets in the QCD β-function where the contributions of Coulomb gluons, transverse gluons and fermions are clearly separated.

The asymptotic freedom is caused by <u>large</u> contribution of the Coulomb gluons to β_{QCD}