Physics of Strong Interactions

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Exercise Sheet 8

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Mass terms in the SU(3) chiral Lagrangian

A mass matrix for the $SU(3)$ chiral Lagrangian is given by the following expression

$$m_{ab}^2 = \frac{2B_0}{F^2} \mathrm{Tr}\left[T^a T^b \hat{M}\right], \tag{1}$$

where T^a, $a = 1..8$ are the Gell-Mann matrices ($\mathrm{Tr}(T^a T^b) = 2\delta^{ab}$) and $\hat{M}$ is the quark mass matrix, $\hat{M} = \mathrm{diag}(m_u, m_d, m_s)$.

1. Prove that m_{ab}^2 does not contain off-diagonal matrix elements except for m_{38}^2 and m_{83}^2.
2. Calculate the mixing angle of π_3 and π_8 and write it in terms of meson masses.

Weak currents and the SU(3) chiral Lagrangian

The weak current that mediates transitions from strange to up quarks can be written as $J_\mu^L = \bar{u}\gamma_\mu(1+\gamma_5)s$. We would like to write this current in terms of Goldstone fields to facilitate calculation of a few amplitudes for weak decays of K-mesons. To do this, we apply the following procedure.

1. We consider a generic left current $J_\mu^{L,a} = \bar{\Psi}_L \gamma_\mu T^a \Psi_L$ where Ψ_L is the quark triplet of left quark fields $\Psi_L = (u_L, d_L, s_L)$ and T^a are the Gell-Mann matrices ($\mathrm{Tr}(T^a T^b) = 2\delta^{ab}$). We add the following term to the QCD Lagrangian
$$\delta\mathcal{L} = B^{a,\mu} J_\mu^{L,a}, \tag{2}$$
where B^a is an auxiliary vector field. We assume that the B-field transforms as $\hat{B}_\mu \to L\hat{B}_\mu L^+$, where $\hat{B} = T^a B^a$, under $SU(3)_L \times SU(3)_R$ transformations. Show that this makes the Lagrangian invariant under $SU(3) \times SU(3)_R$.

2. We now have to construct the Lagrangian that consists of Goldstone fields $\Sigma = e^{i\pi^a T^a/F}$ and the external B-field, has the smallest number of derivatives and is invariant under $SU(3)_L \times SU(3)_R$.

   Show that the generalization of the leading chiral Lagrangian
$$\mathcal{L} = \frac{F^2}{4} \mathrm{Tr}\left[D_\mu \Sigma (D^\mu \Sigma)^+\right] \tag{3}$$
   where $D_\mu = \partial_\mu + iB_\mu$ satisfies these conditions.

3. The left current is given by the functional derivative of the action with respect to B_μ^a, in the limit $B^a \to 0$. Show that this procedure gives the following left current
$$J_\mu^{L,a} = \frac{iF^2}{2} \mathrm{Tr}\left[\Sigma^+ T^a \partial_\mu \Sigma\right]. \tag{4}$$
   To calculate the current as power series in the number of fields, we will use a trick that is described below (see H. Georgi, "Weak interactions and modern particle theory", Sec. 5.7).

4. Show that the following identity identity is valid (hint: expand exponents on both sides in power series)

$$\partial_\mu e^M = \int_0^1 \mathrm{d}s \; e^{sM} \; (\partial_\mu M) \; e^{(1-s)M}. \tag{5}$$

5. Use Eq.(5) in Eq.(4) to show that the left current can be represented by

$$J_\mu^{L,a} = -F \mathrm{Tr}\left[T^a \partial_\mu \hat{\pi}\right] - i \mathrm{Tr}\left[T^a \left[\hat{\pi}, \partial_\mu \hat{\pi}\right]\right] + \frac{2}{3F} \mathrm{Tr}\left[T^a \left[\hat{\pi}, \left[\hat{\pi}, \partial_\mu \hat{\pi}\right]\right]\right] + ... \tag{6}$$

6. Express the weak current $J_\mu^L = \bar{u}\gamma_\mu(1+\gamma_5)s$ in terms of the generic currents $J_\mu^{L,a}$. Write the result for J_μ^L in terms of Goldstone boson fields up to terms that are quadratic in the fields and contain K^-.

7. Use your result for the weak current to compute the matrix elements for semileptonic decay $K^- \to \mu^-\nu$ and $K^- \to \pi^0\mu^-\nu$.

8. The last term in Eq.(6) produces contributions to the current that contain three Goldstone fields. Give examples of K^- decays that such terms can describe.