Physics of Strong Interactions

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Exercise Sheet 11

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The idea of QCD sum rules

The basic idea of QCD sum rules is as follows. Consider a correlator of two currents

$$T_{\mu\nu}(q) = -i \int \mathrm{d}^4 x e^{iqx} \langle 0|TV_\mu(x)V_\nu(0)|0\rangle. \tag{1}$$

We will assume that the currents are conserved $\partial_\mu V^\mu(x) = 0$. In this situation, we can write

$$T_{\mu\nu}(q) = \Pi(q^2)(q^2 g_{\mu\nu} - q_\mu q_\nu). \tag{2}$$

The function $\Pi(q^2)$ satisfies the dispersion relation

$$\Pi(q^2) = \frac{1}{\pi} \int \mathrm{d}s \, \frac{\mathrm{Im}\Pi(s)}{s - q^2}. \tag{3}$$

The idea of QCD sum rules is to compute the two sides of the equation in two different ways: once using the operator product expansion and once using the phenomenological models for the imaginary part of the correlator. We will discuss some elementary steps that are needed to perform these steps.

One can calculate the imaginary part of the function $\Pi(q^2)$ by inserting the sum over all hadron states in Eq.(1). If this is done, the result will be proportional to hadronic matrix elements $\langle 0|V_\mu|X\rangle$. The simplest hadronic states X are single-particle hadronic states that can be produced from a vacuum by a particular current. By choosing the currents we work with, we can study different hadronic states.

1. Consider the currents $V_\mu^{I=1} = 1/2(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)$, $V_\mu^{I=0} = (\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)$, $V_\mu^{(c)} = \bar{c}\gamma_\mu c$. Assuming the isospin symmetry and all other known symmetries of QCD, what are the lighest hadrons that can be produced by these currents from the vacuum? What are their masses, spins, charges, etc.?

2. The imaginary part of the function $\Pi_V(s)$ can be related to the cross section for the e^+e^- annihilation to hadrons with particular quantum numbers. For this, one needs to connect the electromagnetic current with the currents discussed above. Show that

$$\mathrm{Im}\Pi_{V_{I=1}}(s) = \frac{s}{16\pi^2\alpha^2} \sigma^{I=1}_{e^+e^-\to\mathrm{hadrons}}. \tag{4}$$

3. Suppose that $\sigma^{I=1}_{e^+e^-\to\mathrm{hadrons}}$ receives a contribution from a single hadronic state with the mass M_V. Assume that this state has a tiny intrinsic width. Show that

$$\sigma_{e^+e^-\to V} = (2J+1)^2 4\pi^2 \frac{\Gamma_{V\to e^+e^-}}{M_V} \delta(s - M_V^2), \tag{5}$$

   where $\Gamma_{V\to e^+e-}$ is the partial decay width of the resonance V to e^+e^-.

4. We would like to use this expression to compute the correlator $\Pi_{V_{I=1}}(q^2)$, for $q^2 < 0$. To this end, we can consider contributions of all neutral mesons with the isospin one. What are the masses of the first three? If we compute their contributions to Eq.(3), how suppressed are the contibutions of the more massive ones relative to the lightest one?

5. To enhance the contirbution of a ground state and suppress the contribution of the excited states, we can use the so-called Borel transform. The Borel transform of a function is defined as

$$B_M f(Q^2) = \lim_{n\to\infty, Q^2\to\infty, Q^2/n\to M^2=\mathrm{const}} \frac{(Q^2)^{n+1}}{n!}\left(-\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^n f(Q^2). \tag{6}$$

Show that

$$B_M((s+Q^2)^{-1}) = e^{-s/M^2}, \quad B_M(Q^{-2a}) = M^{-2a+2}/(a-1)!, \;\; \mathrm{B}_M(\ln(Q^2)) = -M^2. \tag{7}$$

6. We will now apply the Borel transform to the computation of $\Pi_{I=1}(q^2)$, for $q^2 = -Q^2$, $Q^2 > 0$. First, use the above results to compute the contribution of a resonance of mass M to the Borel transform of the two-point correlator. Under which circumstances does the Borel transform suppress contributions of heavier hadrons to the correlator?

7. Second, imagine that Q^2 is sufficiently large so that $\Pi(Q^2)$ can be computed using the operator product expansion for a particular current in QCD

$$\Pi^{\mathrm{OPE}}_{I=1}(Q^2) = a_1 \ln(Q^2) + a_0 + c_1 \frac{\langle 0|m\bar{q}q|0\rangle}{Q^4} + c_3 \frac{\langle 0|\alpha_s/\pi G^a_{\mu\nu} G^{a,\mu\nu}|0\rangle}{Q^4} + ... \tag{8}$$

Compute the value of the coefficient a_1. The coefficients c_1 and c_3 can be computed using the operator product expansion explained in class; they read

$$c_1 = 2, \quad c_3 = \frac{1}{12} \quad . \tag{9}$$

Perform the Borel transform of the above expression. What can said about the size of the terms neglected in Eq.(8) (ellipses)?

8. Equate the Borel transform of $\Pi^{\mathrm{OPE}}_{I=1}$ and $\Pi^{\mathrm{res}}_{I=1}$, to obtain the sum rule. Explain how it can be used to determine the mass and the leptonic width of the lightest vector $I = 1$ resonance.