Physics of Strong Interactions

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Exercise Sheet 10

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Vector current constraints on the axial anomaly

In deriving the expression for the $SU(3)$ anomaly, it is important to ensure that the vector current is still conserved. We illustrate this with a simpler $U(1)$ example. Consider the variation of the quark action under a $U(1)$ symmetry:

$$\delta S_q = \frac{1}{24\pi^2} \int d^4x (\epsilon_L dA_L \wedge dA_L - \epsilon_R dA_R \wedge dA_R). \tag{1}$$

1. Show that δS_q can be written as

$$\delta S_q = \frac{1}{12\pi^2} \int d^4x \big[2\epsilon dV \wedge dA + c(dA_R \wedge dA_R + dA \wedge dA)\big] \tag{2}$$

   where $\epsilon = (\epsilon_L + \epsilon_R)/2$, $c = (\epsilon_L - \epsilon_R)/2$, $V^\mu = (A_L^\mu + A_R^\mu)/2$ and $A^\mu = (A_L^\mu - A_R^\mu)/2$.

2. The dependence of the above equation on ϵ implies that the vector gauge transformations are anomalous. To correct this, we add an additional term $\delta S_q^{[\mathrm{ct}]}$ such that

$$\delta S_q^{[\mathrm{ct}]} = -\frac{1}{12\pi^2} \int d^4x (2\epsilon dV \wedge dA). \tag{3}$$

   Use integration by parts and the fact that $\partial^\mu \epsilon = \delta_\epsilon V^\mu$ to find the expression for $S_q^{[\mathrm{ct}]}$.

3. Now write $S_q^{\mathrm{full}} = S_q + S_q^{[\mathrm{ct}]}$ and take the variation with respect to c. Compare this with the first two terms of Bardeen's expression for the $SU(3)$ anomaly.

Higher order interactions from the Wess-Zumino-Witten Action

One can expand the WZW action to higher orders, and calculate e.g. the anomalous contribution to the interaction $\gamma\gamma \to \pi^0\pi^0$. We will show that the leading contribution to this process from the WZW action is zero.

1. Show that the expansion of the vector and axial currents to $\mathcal{O}(\pi)$ is

$$\begin{aligned} \hat{v}^\mu(s) =& e\hat{Q}A^\mu + \frac{ie}{F}(1-s)A^\mu \left[\hat{Q}, \hat{\pi}\right] - \frac{i(1-s)^2}{F^2}\hat{\pi}\partial^\mu\hat{\pi} \\ \hat{a}^\mu(s) =& \frac{1-s}{F}\partial^\mu\hat{\pi} \end{aligned} \tag{4}$$

   where $\hat{v}^\mu = (\hat{\ell}^\mu + \hat{r}^\mu)/2 = e\hat{Q}A^\mu$ and $\hat{a}^\mu = (\hat{\ell}^\mu - \hat{r}^\mu)/2 = 0$.

2. Hence, the only term in the WZW action that corresponds to the interaction of two pions and two photons is the first one:

$$f(\hat{c}, \hat{v}^{\mu}, \hat{a}^{\mu}) = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left[3\hat{c}(x)\hat{v}^{\mu\nu}\hat{v}^{\alpha\beta}\right], \tag{5}$$

where $\hat{v}^{\mu\nu} \equiv \partial^{\mu}\hat{v}^{\nu} - \partial^{\nu}\hat{v}^{\mu}$. Furthermore, we can drop the final term in the vector current and write

$$\hat{v}^{\mu}(s) = e\hat{Q}A^{\mu} + \frac{ie}{F}(1-s)A^{\mu}\left[\hat{Q}, \hat{\pi}\right] = e\hat{Q}A^{\mu} + \frac{ie}{F}(1-s)A^{\mu}\left[\hat{Q}, \pi^{b}T^{b}\right] \tag{6}$$

Thus, calculate $f(\hat{\pi}/F, \hat{v}^{\mu}(s), \hat{a}^{\mu}(s)) = f(\pi^{a}T^{a}/F, \hat{v}^{\mu}(s), \hat{a}^{\mu}(s))$ for this interaction, and show that the terms that are relevant for this process are proportional to $\mathrm{Tr}\left(\{\mathrm{T^a}, \hat{\mathrm{Q}}\}[\hat{\mathrm{Q}}, \mathrm{T^b}]\right)$.

3. By evaluating $\mathrm{Tr}\left(\{\mathrm{T^a}, \hat{\mathrm{Q}}\}[\hat{\mathrm{Q}}, \mathrm{T^b}]\right)$, show that the terms in $f(\hat{\pi}/F, \hat{v}^{\mu}(s), \hat{a}^{\mu}(s))$ cannot include two π^0 fields, and hence that the processes $\gamma\gamma \to \pi^0\pi^0$ does not occur at this order.

4. What conclusions can you draw for the process $\gamma\gamma \to \eta^0\eta^0$?