Physics of Strong Interactions

V: Prof. Kirill Melnikov, Ü: Dr. Matthew Dowling

Exercise Sheet 1

Due XX.XX.2015

Gribov Copies

The goal of this exercise is to discuss to what extent the standard "gauge fixing" procedure works in practice. To this end, we consider the Coulomb gauge fixing condition $\partial_i A_i(t,x) = 0$ and imagine that a gauge field $A_i(x)$ that satisfies this condition is found. We assume that this solution is unique and that any gauge transformation applied to A_i produces a field that does not satisfy the Coulomb gauge condition anymore. This is a plausible assumption but in general this is not true. The existence of additional solutions is called the "Gribov anomaly" and the additional solutions themselves are called "Gribov copies". In this exercise we will work through a simple example in $SU(2)$ gauge theory that makes the existence of Gribov copies apparent.

1. Consider an $SU(2)$ field $A_i = \sum A_i^{(a)} \sigma^a/2$ (where $\sigma^{1,2,3}$ are the Pauli matrices) that satisfies the gauge condition $\partial_i A_i = 0$. Perform a gauge transformation $A_i(x) \to A_i^\theta(x)$ in the standard way

$$A_i \to A_i^\theta = U A_i U^{-1} + i g^{-1} (\partial_i U) U^{-1} \tag{1}$$

and show that A_i^θ still satisfies the gauge fixing condition $\partial_i A_i^\theta(x) = 0$ if and only if the following equation is valid

$$\left[D_i, (\partial_i U^{-1}) U\right] = 0, \quad D_i = \partial_i + i g A_i. \tag{2}$$

You may find it useful to employ the following equations

$$U^{-1}\partial_i U = -(\partial_i U^{(-1)})U, \quad \partial_i[(\partial_i U)U^{-1}] = -U[\partial_i(\partial_i U^{-1})U]U^{-1}, \tag{3}$$

that can be obtained by taking derivatives of the two sides of the equation $UU^{-1} = 1$.

2. Show that Eq.(2) can be obtained by computing the extremum of the following functional

$$W = \int d^3\vec{x}\ \mathrm{Tr}\left((\partial_i U)(\partial_i U^{-1}) - 2igA_i(\partial_i U^{-1})U\right) \tag{4}$$

with respect to U. Note that variations of U and of $U^{(-1)}$ are not independent.

3. Clearly, the field $A_i = 0$ satisfies the Coulomb gauge condition. This is a gauge condition that corresponds to the trivial sector of the theory without dynamical fields. We will try to find other gauge potentials that satisfy the Coulomb gauge condition and correspond to the trivial sector of the theory. Indeed, consider a particular gauge transformation matrix U

$$U(\vec{r}) = e^{i\theta(r)\vec{n}\cdot\vec{\sigma}} = \cos(\theta) + i\vec{n}\cdot\vec{\sigma}\sin(\theta), \tag{5}$$

where $\vec{n} = \vec{r}/|\vec{r}|$ and compute the functional W. Minimize it with respect to θ and show that A^θ satisfies the Coulomb gauge if $\theta(r)$ satisfies the following equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\mathrm{d}\theta}{\mathrm{d}t} - \sin(2\theta) = 0, \tag{6}$$

where $t = \ln(r)$.

4. Compute $A_i^\theta(x)$ in terms of $\theta(r)$ and its derivatives using $U(r)$ in Eq.(5). Show that it can be written as
$$A_i^{(a)} = -\frac{2}{g}\left(n_i n_a \frac{\mathrm{d}\theta}{\mathrm{d}r} + (\delta_{ai} - n_a n_i)\,\frac{\sin(2\theta)}{2r} + \epsilon_{aib} n_b \frac{\sin^2\theta}{r}\right). \tag{7}$$

5. By requiring that $A_i^{(a)}$ remain finite at $r \to 0$, determine allowed values of $\theta(0)$. Use this information together with Eq.(6) to find $\theta(r)$ at small values of r.

   One can interpret Eq.(6) as the equation of motion for a particle in the potential $-\sin^2\theta$ subject to an additional friction force. Using this interpretation and the initial condition at $r = 0 (t = -\infty)$ determine the value of $\theta(r = \infty)$ and find the behavior of the solution at large values of r.