Charm and bottom masses at NNLO from electron-positron annihilation at low energies

J.H. Kühn, M. Steinhauser

I Experimental Results for $R$ below $B\bar{B}$-Threshold

$\alpha_s$

II Sum Rules to NNLO with Massive Quarks

$m_Q(m_Q)$

III Summary
I Experimental Results for $R$ below $B\bar{B}$-Threshold

- $\alpha_s$
- data
- $\alpha_s$
Recent “precision” data on $R(s)$
<table>
<thead>
<tr>
<th>experiment</th>
<th>energy [GeV]</th>
<th>date</th>
<th>systematic error</th>
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</thead>
<tbody>
<tr>
<td>BES</td>
<td>2 — 5</td>
<td>2001</td>
<td>4%</td>
</tr>
<tr>
<td>MD-1</td>
<td>7.2 — 10.34</td>
<td>1996</td>
<td>4%</td>
</tr>
<tr>
<td>CLEO</td>
<td>10.52</td>
<td>1998</td>
<td>2%</td>
</tr>
<tr>
<td>PDG $J/\psi$</td>
<td></td>
<td></td>
<td>7%</td>
</tr>
<tr>
<td>PDG $\psi'$</td>
<td></td>
<td></td>
<td>9%</td>
</tr>
<tr>
<td>PDG $\psi''$</td>
<td></td>
<td></td>
<td>15%</td>
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</table>

pQCD and data agree well in the regions

2 — 3.73 GeV and 5 — 10.52 GeV
pQCD includes full $m_Q$-dependence up to $\mathcal{O}(\alpha_s^2)$ and terms of $\mathcal{O}(\alpha_s^3(m^2/s)^n)$ with $n = 0, 1, 2$

can we deduce $\alpha_s$ from the low energy data?

**Result:**

BES below 3.73 GeV: $\alpha_s^{(3)}(3 \text{ GeV}) = 0.369^{+0.047+0.123}_{-0.046-0.130}$

BES at 4.8 GeV: $\alpha_s^{(4)}(4.8 \text{ GeV}) = 0.183^{+0.059+0.053}_{-0.064-0.057}$

MD-1: $\alpha_s^{(4)}(8.9 \text{ GeV}) = 0.193^{+0.017+0.127}_{-0.017-0.107}$

CLEO: $\alpha_s^{(4)}(10.52 \text{ GeV}) = 0.186^{+0.008+0.061}_{-0.008-0.057}$
combined, assuming uncorrelated errors: \[ \alpha_s^{(4)}(5 \text{ GeV}) = 0.235 \pm 0.047 \]

Evolve up to \( M_Z \): \[ \alpha_s^{(5)}(M_Z) = 0.124^{+0.011}_{-0.014} \]

confirmation of running!

result consistent with LEP, but not competitive

(precision of 0.4\% at 3.7 GeV (0.7\% at 2 GeV) would be required)
II. Sum Rules to NNLO with Massive Quarks
   – SVZ Sum Rules, Moments and Tadpoles
   – Tadpoles at Three Loop
   – Results for Charm and Bottom Masses
Some definitions

\[ R(s) = 12\pi \text{Im} \left[ \Pi(q^2 = s + i\epsilon) \right] \]

\[ \left( -q^2 g_{\mu\nu} + q_\mu q_\nu \right) \Pi(q^2) \equiv i \int dx e^{iqx} \langle T j_\mu(x) j_\nu(0) \rangle \]

with the electromagnetic current \( j_\mu \)

Taylor expansion: \( \Pi_c(q^2) = Q_c^2 \frac{3}{16\pi^2} \sum_{n \geq 0} \bar{C}_n \, z^n \)

with \( z = q^2/(4m_c^2) \) and \( m_c = m_c(\mu) \) the \( \overline{\text{MS}} \) mass.

Coefficients \( \bar{C}_n \) up to \( n = 8 \) known analytically in order \( \alpha_s^2 \) (Chetyrkin, JK, Steinhauser)
$\tilde{C}_n$ depend on the charm quark mass through

\[
\begin{align*}
    l_{mc} &\equiv \ln(m_c^2(\mu)/\mu^2) \\
    \tilde{C}_n &= \tilde{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi} \left( \tilde{C}_n^{(10)} + \tilde{C}_n^{(11)} l_{mc} \right) \\
    & \quad + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left( \tilde{C}_n^{(20)} + \tilde{C}_n^{(21)} l_{mc} + \tilde{C}_n^{(22)} l_{mc}^2 \right)
\end{align*}
\]

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<th>3</th>
<th>4</th>
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</thead>
<tbody>
<tr>
<td>$\tilde{C}_n^{(0)}$</td>
<td>1.0667</td>
<td>0.4571</td>
<td>0.2709</td>
<td>0.1847</td>
</tr>
<tr>
<td>$\tilde{C}_n^{(10)}$</td>
<td>2.5547</td>
<td>1.1096</td>
<td>0.5194</td>
<td>0.2031</td>
</tr>
<tr>
<td>$\tilde{C}_n^{(11)}$</td>
<td>2.1333</td>
<td>1.8286</td>
<td>1.6254</td>
<td>1.4776</td>
</tr>
<tr>
<td>$\tilde{C}_n^{(20)}$</td>
<td>2.4967</td>
<td>2.7770</td>
<td>1.6388</td>
<td>0.7956</td>
</tr>
<tr>
<td>$\tilde{C}_n^{(21)}$</td>
<td>3.3130</td>
<td>5.1489</td>
<td>4.7207</td>
<td>3.6440</td>
</tr>
<tr>
<td>$\tilde{C}_n^{(22)}$</td>
<td>$-0.0889$</td>
<td>1.7524</td>
<td>3.1831</td>
<td>4.3713</td>
</tr>
</tbody>
</table>
Define the moments

\[ M_n^{\text{th}} \equiv \frac{12\pi^2}{n!} \left( \frac{d}{dq^2} \right)^n \Pi_c(q^2) \bigg|_{q^2=0} = \frac{9}{4} Q_c^2 \left( \frac{1}{4m_c^2} \right)^n \bar{C}_n \]

dispersion relation:

\[ \Pi_c(q^2) = \frac{q^2}{12\pi^2} \int ds \frac{R_c(s)}{s(s - q^2)} + \text{subtraction} \]

\[ \Rightarrow M_n^{\text{exp}} = \int \frac{ds}{s^{n+1}} R_c(s) \]

\[ M_n^{\text{exp}} = M_n^{\text{th}} \]

\[ \Rightarrow m_c \]
SVZ:

$\Pi^n(0)$ can be reliably calculated in pQCD: low $n$:

- fixed order in $\alpha_s$ is sufficient, in particular no resummation of $1/\nu$ - terms from higher orders required
- condensates are unimportant
- pQCD in terms of short distance mass: $m_c(3\text{ GeV}) \Leftrightarrow m_c(m_c)$
- stable expansion: no pole mass or closely related definition (1S.mass, potential-subtracted mass) involved
- moments available in NNLO
all three-loop – one-scale tadpole amplitudes can be calculated with "arbitrary" power of propagators (Broadhurst; Chetyrkin, JK, Steinhauser); FORM-program MATAD (Steinhauser)

Three-loop diagrams contributing to $\Pi_l^{(2)}$ (inner quark massless) and $\Pi_F^{(2)}$ (both quarks with mass $m$).

Purely gluonic contribution to $\Theta(\alpha_s^2)$
input for $R(s)$

- resonances ($J/\psi, \psi', \psi''$)
- continuum below 4.8 GeV (BES)
- continuum above 4.8 GeV (theory)

experimental error of the moments dominated by resonances

<table>
<thead>
<tr>
<th>$n$</th>
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<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_c(3 \text{ GeV})$</td>
<td>1.027(30)</td>
<td>0.994(37)</td>
<td>0.961(59)</td>
<td>0.997(67)</td>
</tr>
<tr>
<td>$m_c(m_c)$</td>
<td>1.304(27)</td>
<td>1.274(34)</td>
<td>1.244(54)</td>
<td>1.277(62)</td>
</tr>
</tbody>
</table>

error in $m_c$ dominated by experiment for $n=1$,
by theory (variation of $\mu, \alpha_s$) for $n = 3, 4, \ldots$
stability: compare LO, NLO, NNLO \Rightarrow \text{clear improvement}

$m_c(m_c)$ for $n = 1, 2, 3, 4$ in LO, NLO, NNLO.

FINAL RESULT: $m_c(m_c) = 1.304(27)$ GeV
conversion to pole mass (drastic shift from NLO to NNLO!):

\[ M_c^{(2\text{-}\text{loop})} = 1.514(34) \text{ GeV} \]
\[ M_c^{(3\text{-}\text{loop})} = 1.691(35) \text{ GeV} \]

MS-mass more stable

other results from sum rules:
1.23 ± 0.09 GeV Eidemüller, Jamin
1.37 ± 0.09 GeV Penarrocha, Schilcher
1.10 ± 0.04 GeV Narison
1.275 ± 0.015 GeV Ioffe, Zyablyuk

results from lattice simulations:
1.26(4)(12) GeV Becirevic, Lubicz, Martinelli
1.314(40)(20)(7) GeV Rolf, Sint

effect of “perturbative quenching”: ±30 MeV!
Similar analysis for the **bottom quark**: resonances include $\Upsilon(1)$ up to $\Upsilon(6)$, “continuum” starts at 11.2 GeV

\[ m_b(m_b) \text{ for } n = 1, 2, 3 \text{ and } 4 \text{ in LO, NLO and NNLO} \]
RESULT

<table>
<thead>
<tr>
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<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_b(10 \text{ GeV})$</td>
<td>3.665(60)</td>
<td>3.651(52)</td>
<td>3.641(48)</td>
<td>3.655(77)</td>
</tr>
<tr>
<td>$m_b(m_b)$</td>
<td>4.205(58)</td>
<td>4.191(51)</td>
<td>4.181(47)</td>
<td>4.195(75)</td>
</tr>
</tbody>
</table>

$m_b(m_b) = 4.19(5) \text{ GeV}$

corresponds to a pole mass of

$$M_b^{(2-\text{loop})} = 4.63(6) \text{ GeV}$$

$$M_b^{(3-\text{loop})} = 4.80(6) \text{ GeV}$$

consistent with analysis of bottomonium through moments with large $n$

- direct determination of $m_{\text{pole}}$ less stable!
III SUMMARY

\[ \alpha_s^{(4)}(5 \text{ GeV}) = 0.235 \pm 0.047 \]

\[ \Rightarrow \alpha_s^{(5)}(M_Z) = 0.124^{+0.011}_{-0.014} \]

\[ \Rightarrow \text{drastic improvement in } \delta m_c \text{ and } \delta m_b \text{ from moments with low } n \text{ in NNLO} \]

\[ \Rightarrow \text{direct determination of short-distance mass} \]

\[ m_c(m_c) = 1.304(27) \text{ GeV} \]
\[ m_b(m_b) = 4.19(5) \text{ GeV} \]

\[ \text{improved measurements of the cross section in the charm region } (J/\psi !) \text{ would help} \]